Generalized Kurzweil-Stieltjes integrals

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Abstract

We study a scale of generalized Kurzweil-Stieltjes integrals on the real line motivated by Malý and Kuncová (2019). Our generalization is based on the concept of p-oscillation instead of ordinary oscillation, which was the key concept in their definition. Since Kurzweil-Stieltjes integral can be equivalently defined using ordinary oscillation, our improvement consists in using p-oscillation instead of ordinary one and α-systems instead of partitions of the interval. These changes lead to a larger classes of integrable functions. The key concepts like p-oscillation and p-median are discussed in detail. The integrals introduced are non-absolutely convergent and cover both the Lebesgue and Henstock-Kurzweil integral. Our main results concern e.g. the uniqueness of the indefinite generalized integrals and the dependence of the classes of integrable functions on the parameters $p \in [1, \infty)$ and $\alpha \geq 1$. At the end, an analogue of the known Hake's theorem is presented.

1 Introduction

The main topic of this paper is study of the class of HKS^p_α integrals based on minimization of sums of p-oscillations instead of ordinary oscillations and α-systems instead of partitions. Our main results are contained in the closing section 5.

In section 3 we introduce the definition of p-median of a measurable function as a minimizer of its L^p-norm which is the key concept in the definition of p-oscillation and generalized Kurzweil integral. However, it is not obvious how to compute p-oscillation of a given function. It leads us to question if it is possible to classify p-medians of a given function for any p, i.e. how to find a number $c(p) \in \mathbb{R}$ for given measurable function f such that the value of the norm $\|f - c(p)\|_p$ will be minimal. We can answer this question for $p = 1$, $p = 2$ and $p = \infty$. In particular, the p-median for $p = 1$ is the classic median, for $p = 2$ it is the integral mean value and for $p = \infty$ it is the arithmetic mean of essential suprema and infima. Furthermore, if $f \in L^\infty(I)$, then the relations

$$\operatorname*{ess\,inf}_{x \in I} f \leq c(p) \leq \operatorname*{ess\,sup}_{x \in I} f$$

hold for all $p \in [1, \infty]$. We have proved the existence and uniqueness of p-median for $p > 1$. Since L^p is uniformly convex Banach space for $p \in (1, \infty)$ and constant functions its closed convex subset, p-median is the unique "nearest" constant function from the given measurable function $f \in L^p$. Further, we have proved that median of a measurable function on bounded interval always exist but doesn't have to be unique. However, median of continuous function is determined uniquely.

In section 4 we are dealing with the concepts of p-oscillation and the ordinary oscillation. We have proved that both oscillations are pseudonorm, i.e. the functional

$$f \mapsto \operatorname{osc}_p(f,\, I)$$

is a pseudonorm on $L^p(I)$ if $p \in [1, \infty]$ and on the space of bounded functions if $p = C$. Further, we have proved that function $p \mapsto \mathrm{osc}_p(f, I)$ is non-decreasing and if $f \in L^\infty(I)$, then

$$\lim_{p\to\infty} \mathrm{osc}_p(f, I) = \mathrm{osc}_\infty(f, I).$$

For all $p \in [1, \infty]$ and a measurable function $f : I \to \mathbb{R}$ we have proved the relations

$$\mathrm{osc}_p(f, I) \le \mathrm{osc}_\infty(f, I) \le \mathrm{osc}(f, I)$$

which implies the inclusion between HKS^p_α integrable functions

$$HKS_\alpha \subseteq HKS^\infty_\alpha \subseteq HKS^p_\alpha.$$

Generally we can say that features of the ordinary oscillation are more pleasant than features of the p-oscillation. Since p-oscillation neglect Lebesgue null sets, it is not possible to define the definite integral by Newton-Leibniz formula as

$$\int_a^b f \,\mathrm{d}G = F(b) - F(a),$$

which makes a sense only for HKS_α integral. Further, we have proved that ordinary oscillation is subadditive with respect to the interval, i.e. for given function $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$ we have

$$\mathrm{osc}(f, [a, b]) \le \mathrm{osc}(f, [a, c]) + \mathrm{osc}(f, [c, b]),$$

but p-oscillation does not have this property and therefore HKS^p_α is not subadditive with respect to the domain.

In section 5 we introduce the definition of HKS^p_α integral. Recall that it is shown in [8] that the definition of Kurzweil-Stieltjes integral is equivalent to the definition of HKS integral based on the ordinary oscillation. The indefinite HKS_α integral need not to be either continuous or regulated function. But as we have proved, if integrator is continuous, respectively regulated function, then the indefinite integral must be also continuous, respectively regulated, independently on the integrand. Finally, we have proved Hake's theorem for HKS_α integral which says that if F is indefinite HKS_α integral of f with respect to G on $[a, b)$ and if there exist a finite limit

$$L = \lim_{x\to b-} \Big(F(x) - f(b)\,(G(x) - G(b))\Big),$$

then if we define $F(b) := L$, function F will be an indefinite HKS_α integral of f with respect to G on the whole interval $[a, b]$. For related references, see e.g. [1] or [2].

2 Preliminaries

Let M be an arbitrary subset of $\mathbb{R}$, then $\mu(M)$ is the Lebesgue measure of M and, for $p \in [1, \infty]$, $L^p(M)$ is, as usual, the space of real valued functions measurable on M and such that $\|f\|_p < \infty$, where

$$\|f\|_p = \Big(\int_M |f(x)|^p \,\mathrm{d}\,x\Big)^{\frac{1}{p}} \quad \text{if} \quad p \in [1, \infty) \quad \text{and} \quad \|f\|_\infty = \operatorname*{ess\,sup}_{x\in M} |f(x)|$$

is the usual norm on $L^p(M)$.

3 Median and p-median

Next definition was used in [8] (c.f. Definition 2.5 therein) and it is an analogue of median of random variable in probability and statistics, cf. e.g. [10, Section 1.4].

Definition 3.1 (Median)**.** Let $f : [a, b] \to \mathbb{R}$ be a measurable function. We say that the number $\lambda \in \mathbb{R}$ is the *median* of the function f on $[a, b]$ if there exists a measurable set $M \subset [a, b]$ such that $\mu(M) = \frac{1}{2}(b-a)$, $f \le \lambda$ on M and $f \ge \lambda$ on $[a, b] \setminus M$.

Definition 3.2 (p-median)**.** Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^p(I)$ for some $p \in [1, \infty]$. We say that the number $c(p) \in \mathbb{R}$ is the *p-median* of the function f on I if

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \big\|f - c(p)\big\|_p.$$

Remark 3.3. The existence of p-median is obvious. Indeed, since the function $g(c) := \|f - c\|_p$ is non-negative, continuous and its limits at $\pm\infty$ are $+\infty$, it follows that it has a minimum.

As we will prove later, median coincides with p-median for $p = 1$. We will also show that for $p = 2$ the p-median coincides with the integral mean value of the given function, while for $p = \infty$ it is simply the arithmetic mean of essential supremum and infimum of the given function. Furthermore, if $f \in L^\infty(I)$, then the relations

$$\operatorname*{ess\,inf}_{x \in I} f \le c(p) \le \operatorname*{ess\,sup}_{x \in I} f$$

hold for all $p \in [1, \infty]$.

We want to prove the uniqueness of p-median for $p > 1$. First, let us recall the definition of uniformly convex Banach space as used in [3]).

Definition 3.4 (Uniform convexity)**.** We say that a Banach space X is uniformly convex if for every $\varepsilon \in (0, 2]$ there exists a $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$ we have

$$\left\|\frac{x+y}{2}\right\| \le 1 - \delta \quad \text{whenever} \quad \|x - y\| \ge \varepsilon.$$

Remark 3.5. For $p \in (1, \infty)$ is L^p uniformly convex space (Theorem 333, [4]).

Next proposition is taken from [7, Theorem 8.2.2].

Proposition 3.6. *Let X be a uniformly convex Banach space and $Y \subset X$ closed convex subset. Then for every $x \in X$ there exists a unique $y \in Y$ such that*

$$\inf_{z \in Y} \|x - z\| = \big\|x - y\big\|.$$

Proposition 3.7 (Uniqueness of p-median)**.** *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^p(I)$ for some $p \in (1, \infty)$. Then there exist a unique number $c(p) \in \mathbb{R}$ such that*

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \big\|f - c(p)\big\|_p.$$

Proof. Since $X = L^p(I)$ is uniformly convex Banach space and

$$Y := \Big\{f : I \to \mathbb{R};\ f \text{ is constant function a.e. on } I\Big\}$$

is closed convex subset in X, the uniqueness of p-median follows immediately by Proposition 3.6. $\square$

Remark 3.8. The uniqueness of p-median holds also for $p = \infty$, since $c(\infty)$ is the arithmetic mean of essential suprema and infima (as we will see in Proposition 3.14) and this number is determined uniquely.

Now, we will prove that median of a measurable function always exist.

Proposition 3.9. *Every measurable function $f : [a, b] \to \mathbb{R}$ has a median.*

Proof. Let a measurable function $f : [a, b] \to \mathbb{R}$ be given and

$$S_1 := \Big\{\lambda \in \mathbb{R};\ \mu(f^{-1}((-\infty, \lambda))) \le \frac{b-a}{2}\Big\}$$
$$S_2 := \Big\{\lambda \in \mathbb{R};\ \mu(f^{-1}((\lambda, +\infty))) \le \frac{b-a}{2}\Big\}.$$

The monotonicity of measure implies that $h_1(\lambda) := \mu(f^{-1}((-\infty, \lambda)))$ is non-decreasing on $\mathbb{R}$ and $h_2(\lambda) := \mu(f^{-1}((\lambda, +\infty)))$ is non-increasing on $\mathbb{R}$. Moreover, $0 \le h_i(\lambda) \le b-a$ for all $\lambda \in \mathbb{R}$ and $i \in \{1, 2\}$.

Denote $A_k := f^{-1}((-\infty, k))$ for $k \in \mathbb{N}$. Then $A_k \subset A_{k+1}$ for each k and in view of the continuity of measure we get

$$\lim_{k\to\infty} \mu(A_k) = \mu(\bigcup_{k=1}^{\infty} A_k) = b - a.$$

Therefore, there is a $k_1 \in \mathbb{N}$ such that $\mu(f^{-1}((-\infty, k_1))) > \frac{b-a}{2}$. Hence, S_1 is bounded from above. Next, we will show that it is non-empty. To this aim, put $B_k := f^{-1}((-\infty, -k))$ for $k \in \mathbb{N}$. We have $B_{k+1} \subset B_k$ for each k and all these sets have finite measures. Thus, using the continuity of measure again, we obtain

$$\lim_{k\to\infty} \mu(B_k) = \mu(\bigcap_{k=1}^{\infty} B_k) = 0.$$

Therefore, there is a $k_2 \in \mathbb{N}$ such that

$$\mu(f^{-1}((-\infty, -k_2))) < \frac{b-a}{2}.$$

In other words, $S_1 \neq \emptyset$. Analogously, we can prove that S_2 is nonempty and bounded from below.

Obviously $\lambda_1 = \sup S_1 < \infty$ and $-\infty < \lambda_2 = \inf S_2$. Moreover, it is easy to see that $\lambda_2 \le \lambda_1$. Indeed, if the opposite was true, we could find numbers c_1, c_2 such that $\lambda_1 < c_1 < c_2 < \lambda_2$. In such a case we would have

$$\mu(f^{-1}((-\infty, c_1))) > \frac{b-a}{2} \quad \text{and} \quad \mu(f^{-1}((c_2, +\infty))) > \frac{b-a}{2}$$

a contradiction, since $f^{-1}((-\infty, c_1)) \cap f^{-1}((c_2, \infty)) = \emptyset$, while $f^{-1}((-\infty, c_1)) \cup f^{-1}((c_2, \infty)) = [a, b]$.

Let $\lambda_2 < \lambda_1$ and let an arbitrary $\xi \in (\lambda_2, \lambda_1)$ be given. Then we can choose $\xi_1 \in S_1$ and $\xi_2 \in S_2$ in such a way that $\lambda_2 < \xi_2 < \xi < \xi_1 < \lambda_1$. By the definitions of the sets $S_1.S_2$, we have

$$\mu(f^{-1}((-\infty, \xi])) \le \mu(f^{-1}((-\infty, \xi_1))) \le \frac{b-a}{2} \quad \text{and} \quad \mu(f^{-1}((\xi, \infty))) \le \mu(f^{-1}((\xi_2, \infty))) \le \frac{b-a}{2}.$$

Thus,

$$\mu(f^{-1}((-\infty, \xi])) + \mu(f^{-1}((\xi, \infty) \le \mu(f^{-1}((-\infty, \xi_1))) + \mu(f^{-1}((\xi_2, \infty))) \le b - a. \tag{3.1}$$

On the other hand, $f^{-1}((-\infty, \xi]) \cup f^{-1}((\xi, \infty)) = [a, b]$ and this together with (3.1) yields

$$\mu(f^{-1}((-\infty, \xi])) + \mu(f^{-1}((\xi, \infty)) = b - a,$$

i.e. any $\xi \in (\lambda_1, \lambda_2)$ is the median of f.

It remains to consider the case $\lambda_1 = \lambda_2$. Thus, let $\lambda^* := \lambda_1 = \lambda_2$. Then, since $(-\infty, \lambda^*) \subset S_1$, we have $\mu(f^{-1}((-\infty, \lambda))) \le \frac{1}{2}(b-a)$ for all $\lambda < \lambda^*$ and, thanks to the continuity of measure,

$$\mu(f^{-1}((-\infty, \lambda^*))) = \lim_{\lambda\to\lambda^*} \mu(f^{-1}((-\infty, \lambda))) \le \frac{b-a}{2}. \tag{3.2}$$

Similarly,

$$\mu(f^{-1}(\lambda^*, \infty)) \le \frac{b-a}{2}. \tag{3.3}$$

If one of the relations (3.2), (3.3) reduces to the equality, then λ^* will be the median of f. Indeed, if $\mu(f^{-1}(\lambda^*,\infty)) = \frac{b-a}{2}$, then for $M = f^{-1}(\lambda^*,\infty))$, we have $f(M) = (\lambda^*,\infty)$, $\mu(M) = \frac{b-a}{2}$ and $f([a,b]\setminus M) \subset (-\infty,\lambda^*]$.

Now, assume that both inequalities (3.2) and (3.3) are strict. Then, as obviously

$$[a,b] = f^{-1}(((-\infty,\lambda^*))) \cup f^{-1}(\{\lambda^*\}) \cup f^{-1}(((\lambda^*,\infty))),$$

the set $f^{-1}(\{\lambda^*\})$ is nonempty and $\mu(f^{-1}(\{\lambda^*\})) > 0$. We can define $h(t) = \mu([a,t]\cap f^{-1}(\{\lambda^*\}))$ for $t \in [a,b]$. As h is continuous on $[a,b]$, $h(a) = 0$ and $h(b) = b-a$, we can find a $t_0 \in [a,b]$ such that

$$h(t_0) = \mu([a,t_0]\cap f^{-1}(\{\lambda^*\})) = \frac{b-a}{2} - \mu(f^{-1}(-\infty,\lambda^*)) > 0.$$

Furthermore, $f^{-1}(\{\lambda^*\}) = A\cup B$, where $A := [a,t_0]\cap f^{-1}(\{\lambda^*\})$ and $B := (t_0,b]\cap f^{-1}(\{\lambda^*\})$ are disjoint. Simultaneously,

$$\mu(A\cup f^{-1}(-\infty,\lambda^*)) = \frac{b-a}{2}, \quad \mu(B\cup f^{-1}((\lambda^*,\infty))) = \frac{b-a}{2},$$
$$f(x)\le\lambda^* \quad \text{for } x\in A\cup f^{-1}((-\infty,\lambda^*)) \text{ and } f(x)\ge\lambda^* \quad \text{for } x\in B\cup f^{-1}((\lambda^*,\infty)).$$

It follows easily that λ^* is the median of the function f. □

Example 3.10. Median does not have to be uniquely determined, as shown by the following example. The median of the function $f:[0,2]\to\mathbb{R}$ given by the formula

$$f(x) = \begin{cases} 0 & \text{if } x\in[0,1), \\ 1 & \text{if } x\in[1,2] \end{cases}$$

can be any number from the interval $[0,1]$. Indeed, let $M = [0,1)$. Then given an arbitrary $\lambda\in[0,1]$, we have $f\le\lambda$ on M and $f\ge\lambda$ on $[0,2]\setminus M$.

On the other hand, it is easy to verify that, if $p>1$, then all the p-medians of the function f on $[0,2]$ are equal to $\frac{1}{2}$.

Example 3.11. The median of the function $\sin x$ on $[-\pi,\pi]$ is zero, as well as all its p-medians with $p>1$ (shown in [6]).

Example 3.12. The median of the function $\sin x$ on the interval $[0,\pi]$ equals $\frac{1}{2}\sqrt{2}$ because $\sin x\ge\frac{\sqrt{2}}{2}$ for all $x\in[\frac{\pi}{4},\frac{3\pi}{4}]$ and $\sin x\le\frac{\sqrt{2}}{2}$ for all $x\in[0,\frac{\pi}{4})\cup(\frac{3\pi}{4},\pi]$. On the other hand, its p-median $c(\infty)$ for $p=\infty$ equals $\frac{1}{2}$, while $c(2) = \frac{1}{\pi}\int_0^\pi \sin x\,\mathrm{d}x = \frac{2}{\pi}$.

Proposition 3.13. *Median of the continuous function $f:[a,b]\to\mathbb{R}$ is uniquely determined.*

Proof. For a contradiction, let us suppose that f has two medians λ_1,λ_2 such that $\lambda_1<\lambda_2$. then, by the definition of the median, there are measurable sets $M_1, M_2\subset[a,b]$, of measure $\frac{b-a}{2}$ and such that

$$f(x)\le\lambda_1 \text{ for all } x\in M_1 \text{ and } f(x)\ge\lambda_1 \text{ for all } x\in[a,b]\setminus M_1$$

and

$$f(x)\le\lambda_2 \text{ for all } x\in M_2 \text{ and } f(x)\ge\lambda_2 \text{ for all } x\in[a,b]\setminus M_2.$$

Using these properties, we get

$$\frac{b-a}{2} = \mu(M_2) \ge \mu(f^{-1}((-\infty,\lambda_2))) \ge \mu(M_1) + \mu(f^{-1}((\lambda_1,\lambda_2))) = \frac{b-a}{2} + \mu(f^{-1}((\lambda_1,\lambda_2))).$$

It follows that $\mu(f^{-1}((\lambda_1, \lambda_2))) = 0$. However, the preimage of an open interval (λ_1, λ_2) under a continuous mapping f must be an open set. Therefore, $f^{-1}((\lambda_1, \lambda_2))$ is an open set of measure zero, so it must be empty. Thus, the range H_f of the function f must be a subset of the set $[0, \lambda_1] \cup [\lambda_2, \infty]$. Since the continuous image of the interval $[a, b]$ is again an interval, it must be either $H_f \subset [0, \lambda_1]$ or $H_f \subset [\lambda_2, \infty]$. But, in the former case it is $f(x) \leq \lambda_1$ for all $x \in [a, b]$ which implies that λ_2 can not be the median of f. Similarly, in the latter case we have $f(x) \geq \lambda_2$ for all $x \in [a, b]$ which means that λ_1 can not be the median of f. These conclusions contradicts our assumption, of course. □

Proposition 3.14. *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^\infty(I)$. Put*

$$A := \operatorname*{ess\,inf}_{x \in I} f(x) \quad \text{and} \quad B := \operatorname*{ess\,sup}_{x \in I} f(x).$$

Then

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \inf_{c \in [A,B]} \|f - c\|_p \quad \text{for all } p \in [1, \infty].$$

Furthermore,

$$\inf_{c \in \mathbb{R}} \|f - c\|_\infty = \left\| f - \frac{A + B}{2} \right\|_\infty = \frac{B - A}{2}.$$

Proof. (i) First, let us prove the first part of the statement, i.e. that the sought number c will always lie in the interval $[A, B]$. In other words, we want to show that

$$\inf_{z \in [A,B]} \|f - z\|_p \leq \|f - c\|_p \quad \text{for all} \quad c \in (-\infty, A) \cup (B, \infty).$$

If $c \in (B, \infty)$, then for almost all $x \in [a, b]$ we have

$$|f(x) - c| = c - f(x) > B - f(x) = |f(x) - B|.$$

Consequently, $|f(x) - c|^p > |f(x) - B|^p$ for a.e. $x \in I$ and, thus, $\|f - c\|_p \geq \|f - A\|_p$.

In case $p = \infty$ we have

$$\|f - c\|_\infty \geq \|f - B\|_\infty \text{ if } c > B \text{ and } \|f - c\|_\infty \geq \|f - A\|_\infty \text{ if } c < A.$$

To summarize,

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \inf_{c \in [A,B]} \|f - c\|_p \quad \text{for all } p \in [1, \infty].$$

(ii) Let us prove the remaining part of the statement. If $c \in \mathbb{R}$ is an arbitrary constant, then

$$\operatorname*{ess\,inf}_{x \in I} (f(x) - c) = A - c \quad \text{and} \quad \operatorname*{ess\,sup}_{x \in I} (f(x) - c) = B - c.$$

Thus

$$\|f - c\|_\infty = \max\{|A - c|, |B - c|\}.$$

Function $y(c) = \max\{|A - c|, |B - c|\}$ has a minimum for $c = \frac{A+B}{2}$ and therefore

$$\inf_{c \in \mathbb{R}} \|f - c\|_\infty = \max\left\{ \left| A - \frac{A + B}{2} \right|, \left| B - \frac{A + B}{2} \right| \right\} = \frac{B - A}{2}$$

□

Remark 3.15. Analogously, if instead of p-norm we consider the supremum norm $\|f\| = \sup_{x \in I} |f(x)|$, we get

$$\inf_{c \in \mathbb{R}} \|f - c\| = \left\| f - \frac{1}{2} \big(\sup_{x \in I} f(x) + \inf_{x \in I} f(x) \big) \right\|.$$

Proposition 3.16. *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^2(I)$. Then*

$$\inf_{c\in\mathbb{R}} \|f - c\|_2 = \left\| f - \frac{\int_I f(t)\,\mathrm{d}t}{\mu(I)} \right\|_2 .$$

In other words, for $p = 2$ the p-median of f equals to the integral mean value of f.

Proof. Let $c \in \mathbb{R}$. Since $L^2(I) \subset L^1(I)$ for I bounded, both integrals $\int_I f(x)\,\mathrm{d}x$ and $\int_I f^2(x)\,\mathrm{d}x$ exist and are finite. Therefore

$$g(c) := \|f - c\|_2^2 = \int_I \big(f(x) - c\big)^2 \,\mathrm{d}x = \int_I f^2(x)\,\mathrm{d}x - 2c \int_I f(x)\,\mathrm{d}x + c^2 \mu(I).$$

This is a quadratic function of c with a positive leading coefficient and thus it must have a minimum. Its derivative is

$$g'(c) = -2 \int_I f(x)\,\mathrm{d}x + 2c\,\mu(I).$$

Hence

$$g'(c) = 0 \quad \text{if and only if} \quad c = \frac{1}{\mu(I)} \int_I f(x)\,\mathrm{d}x \,.$$

This is its stationary point, and the function g takes a minimum there. Therefore, it is also a minimum of the function $\|f - c\|_2$. □

Example 3.17. Next example shows that even if the p-median $c(p)$ is determined uniquely for all $p \in [1, \infty]$, the function $p \mapsto c(p)$ need not be monotone, in general. Indeed, for the function

$$f(x) = \begin{cases} \sin^2 x & \text{if } x \in [0, \pi], \\ \sin x & \text{if } x \in (\pi, 2\pi] \end{cases}$$

we have $c(1) = c(\infty) = 0$. On the other hand, $c(2)$ is negative, as by Proposition 3.16 we have

$$c(2) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi} \big(\frac{\pi}{2} - 2\big) < 0.$$

Example 3.18. Let $f(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0 & \text{if } x \in I \setminus J, \end{cases}$ where $I \subset \mathbb{R}$ is bounded interval and $J \subset I$ its subinterval. It was shown in [6, Example 2.2.6] that the p-medians $c(p)$ of this function are uniquely determined and

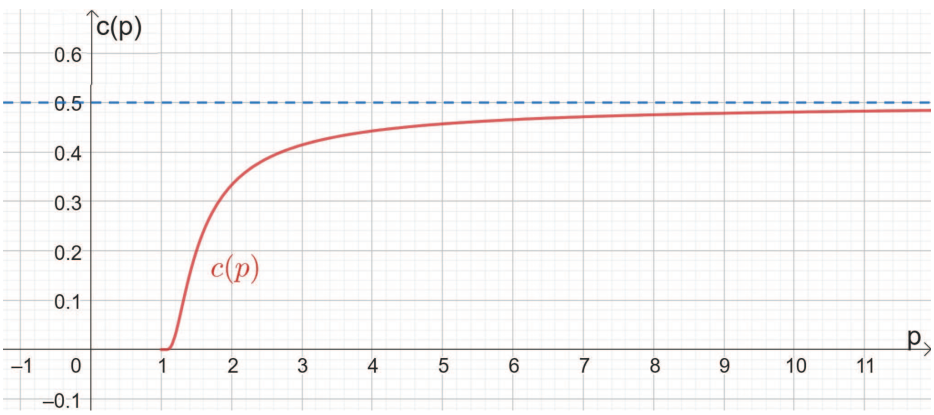

Figure 1: Graph of function $c(p)$ in case $\frac{\mu(I\setminus J)}{\mu(J)} = 2$

they are explicitly given by the formula

$$c(p) = \Big(\big(\frac{\mu(I \setminus J)}{\mu(J)}\big)^{\frac{1}{p-1}} + 1 \Big)^{-1} \quad \text{for } p \in (1, \infty).$$

Notice that the limit of p-medians as $p \to +\infty$ is indeed the arithmetic mean of essential suprema and infima, i.e $\lim_{p \to +\infty} c(p) = \frac{1}{2} = c(\infty)$. Further, notice also that

$$\lim_{p \to 1+} c(p) = 1 = c(1) \;\text{ if } \mu(J) > \mu(I \setminus J) \;\text{ and }\; \lim_{p \to 1+} c(p) = 0 = c(1) \;\text{ if } \mu(J) < \mu(I \setminus J),$$

i.e. the limit of p-medians $c(p)$ as $p \to 1+$ is indeed the median of f.

If $\mu(I \setminus J) = \mu(J)$, then $c(p) = \frac{1}{2}$ for all $p \in (1, \infty]$, while the median of f is not unique at it can be any number from the interval $[0, 1]$. (See Figure 1.)

Example 3.19. Let

$$f(x) = \begin{cases} 8 & \text{if } x \in [0, 1], \\ 0 & \text{if } x \in (1, 6], \\ -4 & \text{if } x \in (6, 10]. \end{cases}$$

Then the p-median $c(p)$ of f on $[0, 10]$ is determined uniquely for any $p \in [1, \infty]$, but no explicit formula for $c(p)$ is available. One can verify that $c(1) = c(3) = 0$, while $c(2) < 0$ and $c(\infty) > 0$. Thus, the function $p \mapsto c(p)$ is not monotone. Notice that the arithmetic mean of suprema and infima is $c(\infty) = 2$, while the integral mean value evaluates $c(2) = -\frac{4}{5}$.

For a given $p > 1$ let us denote $g(c) := \|f - c\|_p^p$. Obviously,

$$g(c) = (8 - c)^p + 5|c|^p + 4(4 + c)^p$$

and g is continuous $[-4, 8]$. Furthermore,

$$g'(c) = \begin{cases} -p\,(8 - c)^{p-1} + 5p\,|c|^{p-1} \operatorname{sgn} c + 4\,p\,(4 + c)^{p-1} & \text{for } \; c \neq 0, \\ -p\,8^{p-1} + p\,4^p & \text{for } \; c = 0. \end{cases}$$

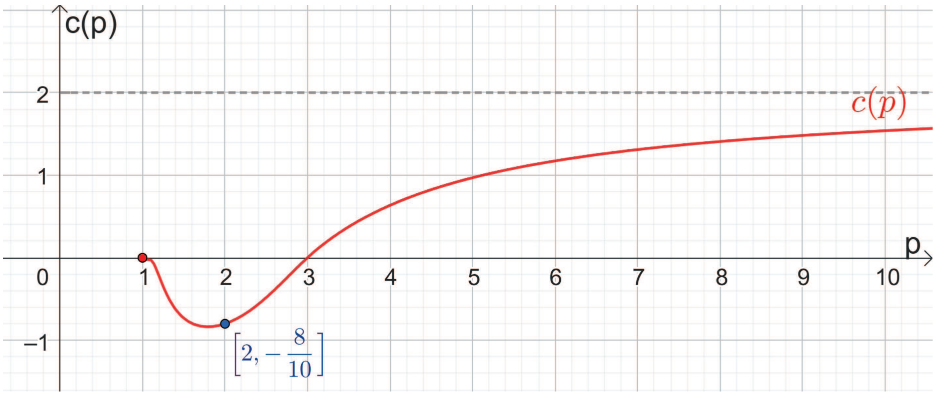

Figure 2: Graph of $c(p)$

One can verify that g' is continuous and increasing on $[-4, 8]$, while

$$g'(-4) < 0,\; g'(8) > 0 \quad \text{and} \quad g'(0-) = g'(0+) = p\,(4^p - 8^{p-1})).$$

In particular, for a given $p \in (1, \infty)$, there is exactly one point $c(p) \in (-4, 8)$ such that $g'(c(p)) = 0$. This defines implicitly the function $p \mapsto c(p)$. In addition, g is decreasing on $[-4, c(p)]$ and increasing on $[c(p), 8]$. Finally, $g'(0) = 0$ if and only if $8^{p-1} = 4^p$, i.e. if and only if $p = 3$, i.e. $c(3) = 0$. (See Figure 2.)

Open problems.

- Limits of p-medians. In Example 5.2 we have seen that for the function f considered there the limit of p-medians $c(p)$ as $p \to \infty$ is $c(\infty)$ and the limit of $c(p)$ as $p \to 1+$ is the median of f. The question is whether this is true in general.

- Properties of p-medians. Is function $p \mapsto c(p)$ continuous? Is it differentiable? Is it true that $\lim\limits_{p \to 1_+} c'(p) = 0$?

- Equivalence of median and minimizer of L^1 norm. We will prove in Proposition 4.4 that any median of f is also p-median for $p = 1$. An interesting question remains whether it is true that a number $\lambda \in \mathbb{R}$ is a median of $f \in L^1(I)$ if and only if

$$\inf_{c \in \mathbb{R}} \|f - c\|_1 = \|f - \lambda\|_1?$$

4 Oscillations and HK^p integral

In this section we introduce the notions of oscillation which is the key concept in the definition of HKS^p integral. Next definition is taken from [8, Definition 2.3].

Definition 4.1 (Oscillations)**.** Let $I \subset \mathbb{R}$ be a bounded interval and $p \in [1, \infty]$. We define the p-oscillation of a measurable function $f : I \to \mathbb{R}$ as

$$\operatorname{osc}_p(f, I) := (\mu(I))^{-\frac{1}{p}} \inf_{c \in \mathbb{R}} \|f - c\|_p.$$

Here and in what follows we set $\frac{1}{p} = 0$ if $p = \infty$.

Definition 4.2. For an arbitrary function $f : I \to \mathbb{R}$ defined on the whole interval I, the value

$$\operatorname{osc}_C(f, I) := \frac{1}{2} \sup_{x,y \in I} |f(x) - f(y)|$$

will be said to be the *ordinary oscillation* of f.

Remark 4.3. We will often skip the symbol C and denote ordinary oscillation as $\operatorname{osc}(f, I)$. Analogously to Malý and Kuncová in [8], we will consider the symbol C as a possible value of p and always put $1/p = 0$ for $p = C$. The subscript C refers to the space of continuous functions and the somewhat unusual factor $\frac{1}{2}$ is an output of the usage of the supremum norm instead of the L^p-norm.

Note that, while in the definition of p-oscillation we require the function f to be measurable, ordinary oscillation can be defined also for non-measurable functions. On the other hand, on the contrary to p-oscillation, the ordinary oscillation does not neglect sets of zero Lebesgue measure.

The ordinary oscillation of the function $f : I \to \mathbb{R}$ can be equivalently defined as

$$\operatorname{osc}(f, I) := \frac{1}{2}\Big(\sup_{x \in I} f(x) - \inf_{x \in I} f(x)\Big).$$

Indeed,

$$\begin{aligned}\sup_{x,y \in I} |f(x) - f(y)| &= \sup_{x,y \in I} (f(x) - f(y)) \\ &= \sup_{x \in I} f(x) + \sup_{y \in I}(-f(y)) = \sup_{x \in I} f(x) - \inf_{y \in I} f(y).\end{aligned}$$

The following proposition is taken from [8, Proposition 2.6].

Proposition 4.4 (Oscillation and median)**.** *Let $\lambda \in \mathbb{R}$ be the median of the function f on the bounded interval $I \subset \mathbb{R}$ and $p \in [1, \infty]$. Then*

$$osc_p(f,\, I) \le (\mu(I))^{-\frac{1}{p}}\, \|f - \lambda\|_p \le 2^{1-\frac{1}{p}}\, osc_p(f,\, I).$$

In particular, for $p = 1$ we get

$$osc_1(f,\, I) = (\mu(I))^{-1}\, \|f - \lambda\|_1$$

and

$$\inf_{c\in\mathbb{R}} \|f - c\|_1 = \|f - \lambda\|_1.$$

(In other words, median coincides with p-median for $p = 1$.)

Next, we will introduce new definition of generalized Kurzweil integral based on minimization of sum of p-oscillations instead of ordinary oscillations which leads to a wider class of integrable functions.

Definition 4.5. We say that $\{[a_i, b_i], x_i\}_{i=1}^n$ ($n \in \mathbb{N}$) is a **tagged partition** of the interval $I \subset \mathbb{R}$ if the intervals $[a_i, b_i]$ are non-overlapping, their union is I and $x_i \in [a_i, b_i]$ for every $i \in \{1, \dots, n\}$.

Definition 4.6. Let an arbitrary positive function $\delta : [a, b] \to \mathbb{R}^+$ be given. We say that the tagged partition $\{[a_i, b_i], x_i\}_{i=1}^n$ is **δ-fine** if

$$[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i)) \quad \text{for all} \;\; i \in \{1, \dots, n\}.$$

Definition 4.7 (Generalized Kurzweil integral)**.** Let $I \subset \mathbb{R}$ be an interval, f, F be functions measurable on I and $p \in [1, \infty]$. We say that F is an indefinite HK^p **integral** of a function f if for all $\varepsilon > 0$ there exists $\delta_\varepsilon : I \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{n} \operatorname{osc}_p(F - f(x_i)\, x, [a_i, b_i]) < \varepsilon \tag{4.1}$$

holds for each δ_ε-fine tagged partition $\{[a_i, b_i], x_i\}_{i=1}^n$ of the interval I.

Remark 4.8. Using ordinary oscillation in (4.1) instead of p-oscillation we get an equivalent definition of an indefinite Kurzweil integral. This follows from [8, Proposition 3.7].

Proposition 4.9. *If f is a measurable function on the bounded interval $I \subset \mathbb{R}$, then*

$$osc_\infty(f,\, I) \le osc(f,\, I).$$

Moreover, if the function f is continuous, then the equality holds.

Proof. From Proposition 3.9 we have

$$\operatorname{osc}_\infty(f,\, I) = \frac{1}{2}\big(\operatorname*{ess\,sup}_{x\in I} f(x) - \operatorname*{ess\,inf}_{x\in I} f(x)\big) \le \frac{1}{2}\big(\sup_{x\in I} f(x) - \inf_{x\in I} f(x)\big) = \operatorname{osc}(f,\, I).$$

Moreover, if the function f is continuous on the interval I, then

$$\inf_{x\in I}\, f(x) = \operatorname*{ess\,inf}_{x\in I} f(x) \quad \text{and} \quad \sup_{x\in I}\, f(x) = \operatorname*{ess\,sup}_{x\in I} f(x),$$

thus the equality of oscillations occurs. □

Example 4.10. For function

$$f(x) = \begin{cases} 0 & \text{if } \; x \in [0, 1] \setminus \{\frac{1}{k};\, k \in \mathbb{N}\}, \\ \frac{1}{x} & \text{if } \; x \in \{\frac{1}{k};\, k \in \mathbb{N}\}. \end{cases}$$

we get $\operatorname{osc}_\infty(f,\, [0, 1]) = 0$, while ordinary oscillation is $\operatorname{osc}(f,\, [0, 1]) = +\infty$.

Example 4.11. Function $F(x) = x + \chi_{\mathbb{Q}}(x)$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the set of rational numbers, is for every $p \in [1, \infty]$ an indefinite HK^p integral of function $f(x) = 1$ on $[0, 1]$. Since p-oscillation neglects values on sets which have Lebesgue measure zero and $F - f(x_i)\, x$ is constant function a.e. for arbitrary tags x_i, we observe

$$\operatorname{osc}_p(F - f(x_i)\, x, [0, 1]) = \operatorname{osc}_p(\chi_{\mathbb{Q}}, [0, 1]) = 0.$$

On the other hand, F is not an indefinite HK integral of f, since ordinary oscillation is

$$\operatorname{osc}(F - f(x_i)\, x, [a_i, b_i]) = \frac{1}{2}$$

for each tagged partition $\{[a_i, b_i], x_i\}_{i=1}^n$ of the interval $[0, 1]$.

Proposition 4.12 (Triangle inequality for oscillation)**.** *If f_1 and f_2 are measurable functions on the bounded interval $I \subset \mathbb{R}$ and $p \in [1, \infty]$ or $p = C$, then*

$$osc_p(f_1 + f_2,\, I) \leq osc_p(f_1,\, I) + osc_p(f_2,\, I).$$

Proof. Let $p \in [1, \infty]$. By Definition4.1 we have

$$\operatorname{osc}_p(f_1 + f_2,\, I) = (\mu(I))^{-\frac{1}{p}} \inf_{c \in \mathbb{R}} \|f_1 + f_2 - c\|_p.$$

Let $c_1, c_2 \in \mathbb{R}$ be such that $c = c_1 + c_2$. Then

$$\begin{aligned} \inf_{c \in \mathbb{R}} \|f_1 + f_2 - c\|_p &= \inf_{c_1, c_2 \in \mathbb{R}} \|f_1 - c_1 + f_2 - c_2\|_p \leq \inf_{c_1, c_2 \in \mathbb{R}} \Big(\|f_1 - c_1\|_p + \|f_2 - c_2\|_p \Big) \\ &= \inf_{c_1 \in \mathbb{R}} \|f_1 - c_1\|_p + \inf_{c_2 \in \mathbb{R}} \|f_2 - c_2\|_p. \end{aligned}$$

Further, we deduce

$$\begin{aligned} \operatorname{osc}_p(f_1 + f_2,\, I) &\leq (\mu(I))^{-\frac{1}{p}} \Big(\inf_{c_1 \in \mathbb{R}} \|f_1 - c_1\|_p + \inf_{c_2 \in \mathbb{R}} \|f_2 - c_2\|_p \Big). \\ &= \operatorname{osc}_p(f_1, I) + \operatorname{osc}_p(f_2,\, I). \end{aligned}$$

For ordinary oscillation we have

$$\begin{aligned} \operatorname{osc}(f_1 + f_2,\, I) &= \frac{1}{2} \sup_{x, y \in I} \Big| (f_1 + f_2)(y) - (f_1 + f_2)(x) \Big| = \frac{1}{2} \sup_{x, y \in I} \Big| f_1(y) + f_2(y) - f_1(x) - f_2(x) \Big| \\ &\leq \frac{1}{2} \sup_{x, y \in I} \Big| f_1(y) - f_1(x) \Big| + \frac{1}{2} \sup_{x, y \in I} \Big| f_2(y) - f_2(x) \Big| = \operatorname{osc}(f_1,\, I) + \operatorname{osc}(f_2,\, I). \end{aligned}$$

□

Remark 4.13. The multiplicative constant $(\mu(I))^{-1/p}$ appearing in the definition of oscillation enables to show that the function $p \to \operatorname{osc}_p(f,\, I)$ is non-decreasing, as we will see in proof of following Proposition.

Proposition 4.14. *Let $I \subset \mathbb{R}$ be a bounded interval and let $f \in L^p(I)$. be measurable. Then the function $H(p) := osc_p(f,\, I)$ is non-decreasing on $[1, \infty)$.*

Proof. Let $p, q \in [1, \infty)$ be such that $p < q$. Since $\frac{p}{q} + \frac{q-p}{q} = 1$, using Hölder's inequality, we obtain

$$\begin{aligned} \int_I |f(x)|^p \, dx &= \int_I \big(|f(x)|^p \, 1 \big) \, \mathrm{d}x \leq \Big(\int_I (|f(x)|^p)^{\frac{q}{p}} \, \mathrm{d}x \Big)^{\frac{p}{q}} \Big(\int_I (1)^{\frac{q}{q-p}} \, \mathrm{d}x \Big)^{\frac{q-p}{q}} \\ &= \Big(\int_I (|f(x)|^p)^{\frac{q}{p}} \, \mathrm{d}x \Big)^{\frac{p}{q}} (\mu(I))^{\frac{q-p}{q}}, \end{aligned}$$

i.e. $\|f\|_p \le \|f\|_q\,(\mu(I))^{\frac{q-p}{pq}}$. Consequently,

$$\begin{aligned} H(p) = \mathrm{osc}_p(f,\, I) &= (\mu(I))^{-\frac{1}{p}} \inf_{c\in\mathbb{R}} \|f-c\|_p \le (\mu(I))^{-\frac{1}{p}} \inf_{c\in\mathbb{R}} \|f-c\|_q\,(\mu(I))^{\frac{q-p}{pq}} \\ &= (\mu(I))^{-\frac{1}{p}} \inf_{c\in\mathbb{R}} \|f-c\|_q\,(\mu(I))^{\frac{1}{p}-\frac{1}{q}} = (\mu(I))^{-\frac{1}{q}} \inf_{c\in\mathbb{R}} \|f-c\|_q = \mathrm{osc}_q(f,\, I) = H(q). \end{aligned}$$

This completes the proof. □

Remark 4.15. Generally, if I is bounded interval and $f \in L^p(I)$ for some $p \in [1,\infty)$, then the function $H(q) = \mathrm{osc}_p(f, I)$ is non-decreasing on $[1, p]$.

Example 4.16. Since $\ln x \in L^p(0,1)$ for every $p \in [1,\infty)$, the function $H(p) = \mathrm{osc}_p(\ln x, (0,1))$ is non-decreasing on $[1,\infty)$, but $\ln x \notin L^\infty(0,1)$.

Example 4.17. Since $x^{-1/2} \in L^p(0,1)$ for every $p \in [1,2)$, the function $H(p) = \mathrm{osc}_p(x^{-1/2}, (0,1))$ is non-decreasing on $[1,2)$.

The following lemma can be found in [9, Theorem 3.10.7].

Lemma 4.18. *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^\infty(I)$, then*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Proposition 4.19. *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^\infty(I)$, then*

$$\lim_{p\to\infty} osc_p(f,\, I) = osc_\infty(f,\, I).$$

Proof. According to Proposition 4.14, the function $H(p) := \mathrm{osc}_p(f,\, I)$ is non-decreasing. Therefore its limit for $p \to \infty$ does exist. Further, notice that by Proposition 3.14, we have

$$\inf_{c\in\mathbb{R}} \|f-c\|_p = \inf_{c\in[A,B]} \|f-c\|_p \quad \text{for } p \in [1,\infty] \quad \text{and} \quad \inf_{c\in\mathbb{R}} \|f-c\|_\infty = \frac{B-A}{2},$$

where

$$A = \operatorname*{ess\,inf}_{x\in I} f(x) \quad \text{and} \quad B = \operatorname*{ess\,sup}_{x\in I} f(x).$$

Since

$$\lim_{p\to\infty} \mathrm{osc}_p(f,\, I) = \lim_{p\to\infty} (\mu(I))^{-\frac{1}{p}} \big(\inf_{c\in[A,B]} \|f-c\|_p \big) = \lim_{p\to\infty} \big(\inf_{c\in[A,B]} \|f-c\|_p \big),$$

we need to prove that

$$\lim_{p\to\infty} \big(\inf_{c\in[A,B]} \|f-c\|_p \big) = \inf_{c\in[A,B]} \|f-c\|_\infty.$$

According to the previous lemma, we have

$$\lim_{p\to\infty} \|f-c\|_p = \|f-c\|_\infty \quad \text{for any} \quad c \in \mathbb{R}. \tag{4.2}$$

We will show that this convergence is uniform with respect to $c \in [A,B]$. To do it, the first step will be the proof that the relation (4.2) is true locally uniformly on $\mathbb{R}$. So, assume that $\varepsilon > 0$ and $c \in \mathbb{R}$ are given and choose $p_0 \in [1,\infty)$ such that

$$\big|\, \|f-c\|_p - \|f-c\|_\infty \big| < \frac{\varepsilon}{2} \quad \text{for all} \quad p \ge p_0.$$

Further, let $\delta > 0$ be such that $\delta\,(1+\mu(I)) < \frac{\varepsilon}{2}$ Then we have

$$\big|\, \|f-c'\|_p - \|f-c'\|_\infty \big| \le \big|\, \|f-c\|_p - \|f-c\|_\infty \big| + |c'-c|\,\mu(I)^{1/p} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $p \geq p_0$ and $c' \in (c-\delta, c+\delta)$. This proves the locally uniform convergence with respect to $c \in \mathbb{R}$ in (4.2). By means of the Vitali Covering Theorem it is now easy to show that $\|f-c\|_p$ tends to $\|f-c\|_\infty$ as $p \to \infty$ uniformly with respect to $c \in [A, B]$.

Now, we will finally prove that

$$\lim_{p\to\infty} \inf_{c\in[A,B]} \|f-c\|_p = \inf_{c\in[A,B]} \|f-c\|_\infty. \tag{4.3}$$

Let us denote $g(c) := \|f-c\|_\infty$, $g_p(c) := \|f-c\|_p$ and let $\varepsilon > 0$ be given. We can find p_0 such that

$$\big|g_p(c) - g(c)\big| < \varepsilon \quad \text{for all } c \in [A, B] \text{ and all } p \geq p_0.$$

Take $c_1, c_2 \in [A, B]$ satisfying

$$g(c_1) < \inf_{c\in[A,B]} g(c) + \varepsilon \quad \text{and} \quad g_p(c_2) < \inf_{c\in[A,B]} g_p(c) + \varepsilon.$$

Then

$$\inf_{c\in[A,B]} g_p(c) \leq g_p(c_1) = g(c_1) + (g_p(c_1) - g(c_1)) \leq \inf_{c\in[A,B]} g(c) + 2\,\varepsilon$$

and

$$\inf_{c\in[A,B]} g(c) \leq g(c_2) = g_p(c_2) + (g(c_2) - g_p(c_2)) \leq \inf g_p(c) + 2\,\varepsilon$$

or

$$\inf_{c\in\mathbb{R}} g(c) - 2\,\varepsilon \leq \inf_{c\in\mathbb{R}} g_p(c) \leq \inf_{c\in\mathbb{R}} g(c) + 2\,\varepsilon$$

for any $p \geq p_0$. This completes the proof of (4.3) and thus the proof of the proposition, as well. □

Proposition 4.20. *Let $I \subset \mathbb{R}$ be a bounded interval, $p \in [1, \infty]$ or $p = C$. Then the functional*

$$f \mapsto osc_p(f,\, I)$$

is a pseudonorm on $L^p(I)$ if $p \in [1, \infty]$ and on the space of bounded functions if $p = C$.

Proof. Triangle inequality was proved in Proposition 4.12. It remains to show that L is homogeneous, i.e

$$\operatorname{osc}_p(t\,f,\, I) = |t|\operatorname{osc}_p(f,\, I) \quad \text{holds for any} \quad t \in \mathbb{R}.$$

In case $p = C$ this relation is obvious. If $p \in [1, \infty]$, then

$$\begin{aligned}\operatorname{osc}_p(t\,f,\, I) &= (\mu(I))^{-\frac{1}{p}} \inf_{c\in\mathbb{R}} \|t\,f - c\|_p = (\mu(I))^{-\frac{1}{p}} \inf_{c\in\mathbb{R}} |t|\, \|f - \frac{c}{t}\|_p \\ &= |t|\, (\mu(I))^{-\frac{1}{p}} \inf_{y\in\mathbb{R}} \|f-y\|_p = |t|\operatorname{osc}_p(f,\, I)\end{aligned}$$

for all $t \neq 0$. For $t = 0$ the desired relation is obviously true. □

Remark 4.21. Note that, since oscillation of every constant function is zero, oscillation can not be a norm.

Lemma 4.22 (Subadditivity of oscillation with respect to the interval)**.** *Let $c \in [a, b]$ and $f : [a, b] \to \mathbb{R}$. Then*

$$osc(f,\, [a, b]) \leq osc(f,\, [a, c]) + osc(f,\, [c, b]). \tag{4.4}$$

Proof. If a function f is unbounded on the interval $[a, b]$, it is also unbounded on at least one of the intervals $[a, c]$ or $[c, b]$. The oscillation of the unbounded function is equal to $+\infty$, so in this case the statement holds trivially.

Assume that the function f is bounded on $[a, b]$ and consider the following cases

- If $\sup_{x\in[a,b]} f(x) = \sup_{x\in[a,c]} f(x)$ and $\inf_{x\in[a,b]} f(x) = \inf_{x\in[a,c]} f(x)$, then

$$\sup_{x,y\in[a,b]} |f(y)-f(x)| = \sup_{x,y\in[a,c]} |f(y)-f(x)| \leq \sup_{x,y\in[a,c]} |f(y)-f(x)| + \sup_{x,y\in[c,b]} |f(y)-f(x)|$$

  Wherefrom our assertion immediately follows.

- If $\sup_{x\in[a,b]} f(x) = \sup_{x\in[c,b]} f(x)$ and $\inf_{x\in[a,b]} f(x) = \inf_{x\in[c,b]} f(x)$, then the proof is analogous to the previous case.

- Let us suppose that $\sup_{x\in[a,b]} f(x) = \sup_{x\in[a,c]} f(x)$ and $\inf_{x\in[a,b]} f(x) = \inf_{x\in[c,b]} f(x)$. We want to prove that $\mathrm{osc}(f,\ [a,b]) \leq \mathrm{osc}(f,\ [a,c]) + \mathrm{osc}(f,\ [c,b])$, i.e.

$$\sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x) \leq \sup_{x\in[a,c]} f(x) - \inf_{x\in[a,c]} f(x) + \sup_{x\in[c,b]} f(x) - \inf_{x\in[c,b]} f(x).$$

  If we subtract $\sup_{x\in[a,b]} f(x)$ from both sides of the inequality and add $\inf_{x\in[a,b]} f(x)$ to both sides of the inequality, we can see that it suffices to prove

$$0 \leq -\inf_{x\in[a,c]} f(x) + \sup_{x\in[c,b]} f(x), \quad \text{or equivalently} \sup_{x\in[c,b]} f(x) \geq \inf_{x\in[a,c]} f(x).$$

  However, this must always be true because $[a,c]\cap[c,b] = \{c\}$ and thus

$$\inf_{x\in[a,c]} f(x) \leq f(c) \leq \sup_{x\in[c,b]} f(x).$$

- In the case $\sup_{x\in[a,b]} f(x) = \sup_{x\in[c,b]} f(x)$ and $\inf_{x\in[a,b]} f(x) = \inf_{x\in[a,c]} f(x)$, the proof is analogous to that of the previous case. □

Remark 4.23. The previous statement holds only for ordinary oscillation (i.e. for $p = C$), but does not hold for p-oscillation whenever $p \in [1,\infty]$. As a counterexample, consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ 1 & \text{if } x \in (1,2]. \end{cases}$$

On the intervals $[0,1]$ and $(1,2]$ it has zero oscillation, since on these intervals it is constant and the p-oscillation neglects sets of zero Lebesgue measure, but on the interval $[0,2]$ the function f has a positive oscillation. Thus

$$\mathrm{osc}_p(f,\ [0,2]) > \mathrm{osc}_p(f,\ [0,1]) + \mathrm{osc}_p(f,\ [1,2]).$$

Remark 4.24. Let $c \in [a,b]$ and $f : [a,b] \to \mathbb{R}$ be a monotone function. Then

$$\mathrm{osc}(f,\ [a,b]) = \mathrm{osc}(f,\ [a,c]) + \mathrm{osc}(f,\ [c,b]).$$

Indeed, let f be non-decreasing on $[a,b]$. Then $\sup_{x\in[a,c]} f(x) = \inf_{x\in[c,b]} f(x)$, $\inf_{x\in[a,b]} f(x) = \inf_{x\in[a,c]} f(x)$ and $\sup_{x\in[a,b]} f(x) = \sup_{x\in[c,b]} f(x)$. Therefore

$$\sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x) = \big(\sup_{x\in[a,c]} f(x) - \inf_{x\in[a,c]} f(x)\big) + \big(\sup_{x\in[c,b]} f(x) - \inf_{x\in[c,b]} f(x)\big).$$

Lemma 4.25. *Let $p \in [1, \infty]$ or $p = C$, J, $I \subset \mathbb{R}$ are bounded intervals and let $f : I \to \mathbb{R}$ be a measurable function. If $J \subset I$, then*

$$osc_p(f,\, J) \le osc_p(f,\, I). \tag{4.5}$$

Proof. For $p = C$ we have

$$\frac{1}{2} \sup_{x,y \in J} \Big| f(y) - f(x) \Big| \le \frac{1}{2} \sup_{x,y \in I} \Big| f(y) - f(x) \Big|.$$

Thus, (4.5) is evidently true.

If $p \in [1, \infty]$, then the inequality

$$\int_J |f(x) - K|^p \,\mathrm{d}x \le \int_I |f(x) - K|^p \,\mathrm{d}x$$

holds for each $K \in \mathbb{R}$. Consequently,

$$\Big(\int_J |f(x) - K|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \le \Big(\int_I |f(x) - K|^p \,\mathrm{d}x \Big)^{\frac{1}{p}}$$

and hence also

$$\inf_{K \in \mathbb{R}} \Big(\int_J |f(x) - K|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \le \inf_{K \in \mathbb{R}} \Big(\int_I |f(x) - K|^p \,\mathrm{d}x \Big)^{\frac{1}{p}}.$$

Thus

$$\mathrm{osc}_p(f,\, J) \le \mathrm{osc}_p(f,\, I).$$

□

Remark 4.26. From the previous two lemmas we immediately obtain that an ordinary oscillation is finite subadditive function on the space of bounded function, i.e. if for given function $f : [a, b] \to \mathbb{R}$ there exists $n \in \mathbb{N}$ such that

$$[a, b] = \bigcup_{i=1}^{n} [a_i, b_i], \quad \text{then} \quad \mathrm{osc}(f,\, \bigcup_{i=1}^{n} [a_i, b_i]) \le \sum_{i=1}^{n} \mathrm{osc}(F,\, [a_i, b_i]).$$

We will use this property of ordinary oscillation in Chapter 5 to prove the additivity of the HKS_α integral with respect to the integration domain (see Theorem 5.13).

Example 4.27. Ordinary oscillation is not countably subadditive, i.e. if $[a, b] = \bigcup\limits_{i=1}^{\infty} [a_i, b_i]$, then, in general, it is not true that

$$\mathrm{osc}(f,\, \bigcup_{i=1}^{\infty} [a_i, b_i]) \le \sum_{i=1}^{\infty} \mathrm{osc}(f,\, [a_i, b_i]) \tag{4.6}$$

holds for each bounded function $f : [a, b] \to \mathbb{R}$. Indeed, consider the function

$$f(x) = \begin{cases} 0 & \text{if } \; x \in [0, 1] \\ 1 & \text{if } \; x \in (1, 2] \end{cases}$$

and the system of intervals $[0, 1]$, $[\frac{3}{2}, 2]$, $[\frac{5}{4}, \frac{3}{2}], \ldots$, i.e.

$$\big\{ [a_0, b_0] = [0, 1], [a_i, b_i] = \Big[1 + \frac{1}{2^{i-1}}, 1 + \frac{1}{2^{i-2}} \Big];\, i \ge 2 \big\}.$$

Then $[0, 2] = \bigcup\limits_{i=0}^{\infty} [a_i, b_i]$. The oscillation of the function f on the interval $[0, 2]$ equals 1, but on each of the intervals $[a_i, b_i]$ the function f is constant and therefore has zero oscillation there. Thus, (4.6) does not hold for this function.

Example 4.28. Let $f(x) = x$ on $I = [-1, 1]$. Then, obviously, $\mathrm{osc}(x, [-1,1]) = 1$ and, since f is continuous on $[-1, 1]$, we have by Proposition 4.9 that $\mathrm{osc}_\infty(x, [-1,1]) = 1$. On the other hand, one can verify that all its p-medians are zero for $p \in [1, \infty]$. Therefore for p-oscillation of function f we get

$$\mathrm{osc}_p(x, [-1,1]) = 2^{-\frac{1}{p}} \inf_{c\in\mathbb{R}} \|x - c\|_p = 2^{-\frac{1}{p}} \|x\|_p = 2^{-\frac{1}{p}} \left(\int_{-1}^{1} |x|^p \,\mathrm{d}x \right)^{\frac{1}{p}} = \left(\frac{1}{p+1} \right)^{\frac{1}{p}}.$$

As we can see, function $p \mapsto \mathrm{osc}_p(f, I)$ is non-decreasing and

$$\lim_{p\to+\infty} \mathrm{osc}_p(f, I) = \lim_{p\to+\infty} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} = \mathrm{osc}_\infty(f, I).$$

Example 4.29. Let $f(x) = \sin x$ on $I = [-\pi, \pi]$. Then

$$\mathrm{osc}(\sin x, [-\pi, \pi]) = \frac{1}{2} \big| \sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2}) \big| = 1.$$

The same holds also for as p-oscillation of f on I if $p = \infty$, since f is continuous on I. One can verify that all its p-medians for $p \in [1, \infty]$ are zero. Therefore,

$$\mathrm{osc}_p(\sin\, x, [-\pi, \pi]) = (2\pi)^{-\frac{1}{p}} \|\sin x\|_p.$$

Using software Mathematica we can verify that

$$\mathrm{osc}_p(\sin x, [-\pi, \pi]) = \pi^{-\frac{1}{2p}} \left(\frac{\Gamma(\frac{p+1}{2})}{\Gamma(1 + \frac{p}{2})} \right)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty].$$

In particular,

$$\mathrm{osc}_1(\sin x, [-\pi, \pi]) = (2\pi)^{-1} \|\sin x\|_1 = \frac{2}{\pi}$$

and

$$\mathrm{osc}_2(\sin x, [-\pi, \pi]) = (2\pi)^{-\frac{1}{2}} \|\sin x\|_2 = \frac{\sqrt{2}}{2}.$$

(See Figure 3.)

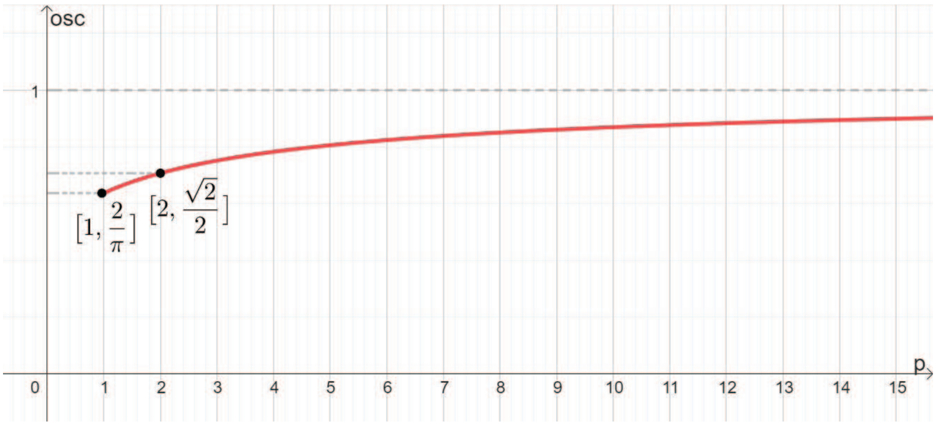

Figure 3: Graph of $p \mapsto \mathrm{osc}_p(\sin x, [-\pi, \pi]) = \pi^{-\frac{1}{2p}} \left(\frac{\Gamma(\frac{p+1}{2})}{\Gamma(1+\frac{p}{2})} \right)^{\frac{1}{p}}$

5 Generalized Kurzweil-Stieltjes HKS_α^p integrals

In this chapter, we introduce the definition of HKS_α^p integral which is generalization of the Kurzweil HK^p integral (introduced in definition 4.7) and deal with its properties.

Definition 5.1 (α-system)**.** Let $\alpha \geq 1$ and $I \subset \mathbb{R}$ be an interval. A system of intervals with tagged points $\{[a_i, b_i], x_i\}_{i=1}^n$ is called a (tagged) α-system in I if the intervals

$$(x_i - \alpha\,(x_i - a_i), x_i + \alpha(b_i - x_i))$$

are subsets of I and pairwise disjoint.

Let $\delta : [a, b] \to \mathbb{R}^+$ be a gauge, we say that the α-system $\{[a_i, b_i], x_i\}_{i=1}^n$ is δ-fine if

$$[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i)) \quad \text{for all} \;\; i \in \{1, \dots, n\}.$$

The following definition of the HKS_α^p integral is a generalization of the HK^p integral in the aspects that instead of tagged partitions we consider tagged α-systems and instead of identity integrators we admit wider class of integrators, like in Stieltjes integrals. For $\alpha = 1$ and $p = C$ this definition is equivalent to the classical definition of the Kurzweil-Stieltjes integral, as we will show in Proposition 5.12.

Definition 5.2 (HKS_α^p integral)**.** Let $I \subset \mathbb{R}$ be an interval, $\alpha \geq 1$, $p \in [1, \infty]$ or $p = C$ and f, F, G be measurable functions on I. We say that F is an indefinite HKS_α^p integral of a function f with respect to a function G if for all $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon : I \to \mathbb{R}^+$ such that the inequality

$$\sum_{i=1}^{n} \mathrm{osc}_p(F - f(x_i)\,G,\, [a_i, b_i]) < \varepsilon \tag{5.1}$$

is true for all δ_ε-fine α-systems $\{[a_i, b_i], x_i\}_{i=1}^n$ in the interval I.

Notation 5.3. By the symbol HK_α^p we will mean HKS_α^p for the choice $G(x) = x$.

Remark 5.4. In the literature one can meet also the notion of so called *centered integrals.* An α-system $\{[a_i, b_i], x_i\}_{i=1}^n$ is said to be centered if every tag x_i is a center of the interval $[a_i, b_i]$ and the HKS_α^p integral is said to be centered if only centered α-systems are considered in Definition 5.2. It is clear from the definition that the existence of the uncentered integral implies the existence of the corresponding centered one.

Remark 5.5. Recall that p-oscillation neglects sets of zero Lebesgue measure. Thus, if $p \in [1, \infty]$, F is the indefinite HKS_α^p integral of f with respect to G on I and Φ vanishes a.e. on I, then $F + \Phi$ is also an indefinite integral of f with respect to G on I. In particular, for $p \in [1, \infty]$ it makes no sense to define a definite integral by the usual Newton-Leibniz formula, i.e. $\int_a^b f\,dG = F(b) - F(a)$. Such a definition of the definite integral makes sense for HKS_α integral, i.e. the case $p = C$.

Remark 5.6. If $\alpha_1 < \alpha_2$, then every α_2-system in I is also a α_1 system. Hence, for any fixed $p \in [1, \infty]$, the class of HKS_α^p integrable functions increases or remains the same as α increases. It is worth mentioning that it was shown in [8] that the classes of HK_α integrable functions are identical for $\alpha \in [1, 2]$ and increasing for $\alpha > 2$.

5.1 Properties of HKS_α^p integrals

Proposition 5.7. *Let $\alpha \geq 1$ and $p, q \in [1, \infty]$. For a given integrator $G : [a, b] \to \mathbb{R}$, let us denote by HKS_α^p the set of all HKS_α^p integrable functions on the interval $[a, b]$ with respect to G. Then*

$$HKS_\alpha^q \subseteq HKS_\alpha^p \quad \text{if} \;\; p < q.$$

Proof. Let us suppose that the function F is the indefinite HKS^q_α integral function f with respect to G on $[a, b]$. By Definition 5.2, this means that

$$\text{for all } \ \varepsilon > 0 \ \text{ there is a gauge } \delta_\varepsilon \text{ such that the inequality } \ \sum_{i=1}^{n} \operatorname{osc}_q(F - f(x_i)\, G,\, [a_i, b_i]) < \varepsilon \ \text{ holds.}$$

From Proposition 4.14, we know that the function $H(q) := \operatorname{osc}_q(f,\, I)$ is non-decreasing, thus

$$\operatorname{osc}_p(f,\, I) \le \operatorname{osc}_q(f,\, I) \quad \text{if} \ \ p \le q.$$

Hence

$$\sum_{i=1}^{n} \operatorname{osc}_p(F - f(x_i)G,\, [a_i, b_i]) \le \sum_{i=1}^{n} \operatorname{osc}_q(F - f(x_i)G,\, [a_i, b_i]) < \varepsilon,$$

which implies that F is an indefinite HKS^p_α integral of the function f with respect to G on $[a, b]$, as well. ☐

Remark 5.8. By [8, Theorem 5.5], the inclusion in the previous proposition is actually strict, i.e., if $p < q$, then $HKS^q_\alpha \subsetneq HKS^p_\alpha$.

Proposition 5.9. *Let $\alpha \ge 1$, and $f,\, G : I \to \mathbb{R}$. Then any indefinite HKS^C_α integral of f with respect to G is also indefinite HKS^∞_α integral of f with respect to G.*

Proof. Let us suppose that F is an indefinite HKS^C_α integral of f with respect to G on I. Let $\varepsilon > 0$ be given and let δ_ε be a gauge such that

$$\sum_{i=1}^{n} \operatorname{osc}(F - f(x_i)\, G,\, [a_i, b_i]) < \varepsilon \quad \text{for each} \ \ \delta_\varepsilon - \text{fine} \ \ \alpha - \text{system} \ \ \{[a_i, b_i], x_i\}_{i=1}^n.$$

From 4.9 we know that the inequality $\operatorname{osc}_\infty(h,\, J) \le \operatorname{osc}(h,\, J)$ holds for any interval J and any measurable function $h : J \to \mathbb{R}$. Therefore

$$\sum_{i=1}^{n} \operatorname{osc}_\infty(F - f(x_i)\, G,\, [a_i, b_i]) \le \sum_{i=1}^{n} \operatorname{osc}(F - f(x_i)\, G,\, [a_i, b_i]) < \varepsilon,$$

for all δ_ε-fine α-system $\{[a_i, b_i], x_i\}_{i=1}^n$ and, thus, F is an indefinite HKS^∞_α integral of f with respect to G on I, as well. ☐

The proof of the following statement can be found in [8, Theorem 3.5].

Proposition 5.10 (Uniqueness of the HKS^p_α integral)**.** *Let $I \subset \mathbb{R}$ be an open interval, $\alpha \ge 1$, $p \in [1, +\infty]$ and let $F_1,\, F_2 : I \to \mathbb{R}$ be the indefinite HKS^p_α integrals of the function f with respect to the function G on I. Then there exists a constant $C \in \mathbb{R}$ such that $F_1 - F_2 = C$ almost everywhere.*

The proof of the following natural assertion follows immediately from the fact that if J is a subinterval of the interval I, then every δ_ε-fine α-system in J is also δ_ε-fine α-system in I.

Proposition 5.11 (Integral on a subinterval)**.** *Let F be the indefinite HKS^p_α integral of the function f with respect to G on the interval I. If $J \subset I$ is an arbitrary subinterval, then F is also the indefinite integral of the function f with respect to G on the interval J.*

5.2 Properties of HKS_α integrals

In this subsection, we deal with the properties of the HKS^p_α integral for $p = C$. The following statement discloses the relationship between the Henstock-Kurzweil-Stieltjes integral and the integrals of the class HKS_α. It is taken from [8, Proposition 3.7].

Proposition 5.12. *Let $I \subset \mathbb{R}$ be an open interval and let $f, F, G : I \to \mathbb{R}$ be measurable functions. Then F is the indefinite HKS integral of the function f with respect to G on I if and only if F is the indefinite HKS_1 integral of the function f with respect to G on I.*

Proposition 5.13 (Additivity of the HKS_α integral with respect to the domain of integration)**.** *Let $f, F, G : [a, b] \to \mathbb{R}$ be measurable functions. If F is the indefinite HKS_α integral of f with respect to G on the interval $[a, c]$ and on the interval $[c, b]$, then it is also indefinite integral of f with respect to G on $[a, b]$.*

Proof. For a given $\varepsilon > 0$, we find the gauges $\delta'_\varepsilon : [a, c] \to \mathbb{R}^+$ and $\delta''_\varepsilon : [c, b] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{n} \operatorname{osc}\Big(F - f(x_i)\, G,\ [a_i, c_i]\Big) < \frac{\varepsilon}{2} \quad \text{for all } \ \delta'_\varepsilon\text{-fine } \alpha\text{-systems}\{[a_i, c_i], x_i\}_{i=1}^n \text{ in } [a, c]$$

and

$$\sum_{j=1}^{m} \operatorname{osc}\Big(F - f(y_j)\, G,\ [c_j, b_j]\Big) < \frac{\varepsilon}{2} \quad \text{for all } \ \delta''_\varepsilon\text{-fine } \alpha\text{-systems}\{[c_i, b_i], y_i\}_{i=1}^m \text{ in } [c, b].$$

Put

$$\delta_\varepsilon(x) := \begin{cases} \min\{\delta'_\varepsilon(x), \frac{1}{2}(c - x)\} & \text{if } \ x \in [a, c), \\ \min\{\delta'_\varepsilon(c), \delta''_\varepsilon(c)\} & \text{if } \ x = c, \\ \min\{\delta''_\varepsilon(x), \frac{1}{2}(x - c)\} & \text{if } \ x \in (c, b] \end{cases}$$

and let $\{[A_i, B_i], z_i\}_{i=1}^n$ be an arbitrary δ_ε-fine α-system in $[a, b]$. In view of the construction of the gauge δ_ε we can see that

$$x + \delta_\varepsilon(x) \le x + \frac{1}{2}(c - x) < c \quad \text{if } \ x < c,$$

and

$$x - \delta_\varepsilon(x) \ge x - \frac{1}{2}(x - c) > c \quad \text{if } \ x > c$$

and therefore if $x \neq c$, then necessarily $c \notin (x - \delta_\varepsilon(x), x + \delta_\varepsilon(x))$. Consequently, either c does not belong to any of the intervals $[A_i, B_i]$ or $c = z_m$ for some $m \in \{1, ..., n\}$. So suppose that such an interval exists in this system. Then

$$\sum_{i=1}^{n} \operatorname{osc}(F - f(z_i)\, G, [A_i, B_i]) = \sum_{i=1}^{m-1} \operatorname{osc}(F - f(z_i)\, G, [A_i, B_i]) + \operatorname{osc}(F - f(c)\, G, [A_m, B_m]) + \sum_{i=m+1}^{n} \operatorname{osc}(F - f(z_i)\, G, [A_i, B_i]).$$

By Lemma 4.22 we have

$$\operatorname{osc}(F - f(c)\, G, [A_m, B_m]) \le \operatorname{osc}(F - f(c)\, G, [A_m, c]) + \operatorname{osc}(F - f(c)\, G, [c, B_m]).$$

Hence,

$$\sum_{i=1}^{n} \operatorname{osc}(F - f(z_i)\, G, [A_i, B_i]) \le \sum_{i=1}^{m-1} \operatorname{osc}(F - f(z_i)\, G, [A_i, B_i]) + \operatorname{osc}(F - f(c)\, G, [A_m, c])$$
$$+ \operatorname{osc}(F - f(c)\, G, [c, B_m]) + \sum_{i=m+1}^{n} \operatorname{osc}(F - f(z_i)\, G, [A_i, B_i]) < \varepsilon.$$

Thus, F is the indefinite integral of f with respect to G on the interval $[a, b]$. The modification of the proof in the case when c does not belong to any of the intervals $[A_i, B_i]$, $i \in \{1, \dots, n$, is obvious. This completes the proof. □

Example 5.14. The statement of the previous proposition does not hold for $p \in [1, \infty]$. A counterexample is given by $f \equiv 0$ on $[0, 2]$ and an arbitrary $G : [0, 2] \to \mathbb{R}$ and $p = 1$. Then the function

$$F(x) = \begin{cases} 0 & \text{if } \; x \in [0, 1), \\ 1 & \text{if } \; x \in [1, 2], \end{cases}$$

is an indefinite HKS^p_α integral of f with respect to G both on $[0, 1]$ and $[1, 2]$, but it is not its indefinite HKS^p_α integral on $[0, 2]$ for $p \in [1, \infty]$. Indeed, choose $\varepsilon \in (0, \frac{1}{2})$ and consider an arbitrary gauge $\delta : [0, 2] \to (0, 2)$. Take a δ-fine α-system $\{[a_1, b_1], x_1\}$ formed by the single interval $[a_1, b_1] = [1 - \delta/2, 1 + \delta(1)/2] \subseteq [0, 2]$ and the tag $x_1 = 1$. The length of this interval equals $\delta(1)$. By Example 3.10, the median of the function F on $[1 - \delta/2, 1 + \delta/2]$ can be any number from the interval $[0, 1]$. In particular, by Proposition 4.4, we have for $p = 1$

$$\operatorname{osc}_1\big(F - f(1)\, G, [1 - \delta/2, 1 + \delta/2]\big)$$
$$= \operatorname{osc}_1\big(F, \big[1 - \delta/2, 1 + \delta/2\big]\big) = \frac{1}{\delta(1)} \int_{1-\delta/2}^{1+\delta/2} F(x)\, \mathrm{d}x = \frac{1}{\delta(1)} \int_{1}^{1+\delta/2} 1\, \mathrm{d}x = \frac{1}{2} > \varepsilon.$$

Thus, F is can not be an indefinite HKS^1_α integral of the function f on $[0, 2]$. Therefore, according to Proposition 5.7, it can not be HKS^p_α integral on $[0, 2]$ for any $p \in [1, \infty]$.

Proposition 5.15 (Uniqueness of the HKS_1 integral)**.** *Let $J \subset \mathbb{R}$ be an interval and let F_1, $F_2 : J \to \mathbb{R}$ be the indefinite HKS_1 integrals of the function f with respect to the function G on the interval J. Then there exists a constant $C \in \mathbb{R}$ such that $F_1 = F_2 + C$ on J.*

Proof. It suffices to prove that $F_1 - F_2$ is constant on any closed subinterval $I \subset J$. Let $\varepsilon > 0$ be given. We can find gauges δ'_ε, δ''_ε on I such that

$$\sum_{i=1}^{n} \operatorname{osc}\big(F_1 - f(z_i)\, G, [c_i, d_i]\big) < \frac{\varepsilon}{2} \quad \text{for each } \; \delta'_\varepsilon\text{-fine system } \; \{[c_i, d_i], z_i\}_{i=1}^{n} \text{ in } I$$
$$\sum_{i=1}^{m} \operatorname{osc}\big(F_2 - f(t_i)\, G, [u_i, v_i]\big) < \frac{\varepsilon}{2} \quad \text{for each } \; \delta''_\varepsilon\text{-fine system } \; \{[u_i, v_i], t_i\}_{i=1}^{m} \text{ in } I.$$

Take

$$\delta_\varepsilon(x) = \min(\delta'_\varepsilon(x), \delta''_\varepsilon(x)) \quad \text{for all } \; x \in I.$$

Then every δ_ε-fine system in I is both δ'_ε-fine and δ''_ε-fine. Now, consider an arbitrary δ_ε-fine tagged partition $\{[a_i, b_i], x_i\}_{i=1}^{n}$, i.e., a system covering the interval I. According to Cousin's lemma, such a system must exist. Then, making use of subadditivity of oscillation with respect to interval (see Lemma 4.22) and triangle inequality for oscillation (see Proposition 4.12), we get

$$\begin{aligned}\operatorname{osc}(F_1-F_2,I) &\le \sum_{i=1}^{n}\operatorname{osc}\big(F_1-F_2,[a_i,b_i]\big)=\sum_{i=1}^{n}\operatorname{osc}\big(F_1-f(x_i)\,G+(-F_2+f(x_i)\,G),[a_i,b_i]\big)\\ &\le \sum_{i=1}^{n}\operatorname{osc}\big(F_1-f(x_i)\,G,[a_i,b_i]\big)+\sum_{i=1}^{n}\operatorname{osc}\big(F_2-f(x_i)\,G,[a_i,b_i]\big)<\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ can be chosen arbitrarily small, it follows that $\operatorname{osc}(F_1-F_2,I)=0$. On the other hand, since

$$\operatorname{osc}(F_1-F_2,I)=\frac{1}{2}\Big(\sup_{x\in I}(F_1(x)-F_2(x))-\inf_{x\in I}(F_1(x)-F_2(x))\Big)$$

(see Remark 4.3), it is clear that F_1-F_2 must be constant on I. □

Proposition 5.16. *Let $\alpha\ge 1$, let F be the indefinite HKS_α integral of the function f with respect to G on $[a,b]$ and let $c\in[a,b]$. Then*

$$\lim_{x\to c}\Big(F(x)+f(c)\,(G(c)-G(x))\Big)=F(c). \tag{5.2}$$

(If c is an endpoint of the interval $[a,b]$, we consider the corresponding one-sided limit.)

Proof. For a given $\varepsilon>0$, we find the gauge δ_ε such that

$$\sum_{i=1}^{n}\operatorname{osc}(F-f(x_i)\,G,\ [a_i,b_i])<\varepsilon.$$

holds for all δ_ε-fine α-systems $\{[a_i,b_i],x_i\}_{i=1}^n$ in $[a,b]$.

Let $c<b$. For any $x\in(c,c+\delta_\varepsilon(c))$ such that $[c,c+\alpha\,(x-c))\subset[a,b]$ consider a δ_ε-fine α-system $\{[c,x],c\}$ formed by one interval. From the definition of the integral we have

$$\operatorname{osc}\big(F-f(c)\,G,[c,x]\big)=\frac{1}{2}\Big(\sup_{t\in[c,x]}(F(t)-f(c)\,G(t))-\inf_{t\in[c,x]}(F(t)-f(c)\,G(t))\Big)<\varepsilon.$$

In particular,

$$\big|F(x)-F(c)-f(c)\,(G(x)-G(c))\big|<2\varepsilon. \tag{5.3}$$

Analogously, if $c>a$, we have

$$\operatorname{osc}\big(F-f(c)\,G,[x,c]\big)=\frac{1}{2}\big(\sup_{t\in[x,c]}(F(t)-f(c)\,G(t))-\inf_{t\in[x,c]}(F(t)-f(c)\,G(t))\big)<\varepsilon$$

for any $x\in(c-\delta_\varepsilon(c),c)$ satisfying $(c-\alpha\,(c-x),c]\subset[a,b]$ and a δ_ε-fine α-system $\{[x,c],c\}$ formed by the interval. Thus,

$$\Big|F(x)-F(c)-f(c)(G(x)-G(c))\Big|<2\varepsilon. \tag{5.4}$$

From estimates (5.3) and (5.4) we obtain that the inequality

$$\Big|F(x)-F(c)+f(c)\,(G(c)-G(x))\Big|<2\varepsilon$$

is true for all $x\in(c-\delta_\varepsilon(c),c+\delta_\varepsilon(c))\cap[a,b]$. □

Definition 5.17 (Regulated function)**.** We say that a function $f : [a, b] \to \mathbb{R}$ is regulated if it has finite one-sided limits at every point of the interval $[a, b]$, i.e. the limits

$$f(t+) = \lim_{\tau \to t+} f(\tau) \quad \text{and} \quad f(s-) = \lim_{\sigma \to s-} f(\sigma)$$

exist in $\mathbb{R}$ for all $t \in [a, b)$ and $s \in (a, b]$, respectively.

Next statement is an obvious corollary of Proposition 5.16

Corollary 5.18. *Let F be the indefinite HKS_α integral of f with respect to G on the interval $[a, b]$. If G is a regulated function, then F is also a regulated function and*

$$\lim_{x \to c+} F(x) = F(c) + f(c) \Big(\lim_{x \to c+} G(x) - G(c) \Big) \in \mathbb{R} \quad \textit{for all} \;\; c \in [a, b) \tag{5.5}$$

and

$$\lim_{x \to c-} F(x) = F(c) - f(c) \Big(G(c) - \lim_{x \to c-} G(x) \Big) \in \mathbb{R} \quad \textit{for all} \;\; c \in (a, b]. \tag{5.6}$$

In particular, if G is continuous on $[a, b]$, then the indefinite integral F is also continuous on $[a, b]$.

Proof. Let G be regulated on $[a, b]$ and $c \in [a, b)$. Then, by Proposition 5.16, we get

$$F(c) = \lim_{x \to c+} \big(F(x) + f(c) \left(G(c) - G(x) \right) \big) = \lim_{x \to c+} F(x) + f(c) \big(G(c) - \lim_{x \to c+} G(x) \big),$$

i.e. (5.5) is true. The proof of (5.6) is quite analogous. The assertion concerning continuity is the an immediate consequence. □

Remark 5.19. Notice that by Corollary 5.18 any indefinite HK_α integral is continuous.

Proposition 5.20 (Uniqueness of the HKS_α integral)**.** *Let $J \subset \mathbb{R}$ be an open interval, $\alpha > 1$ and let F_1, $F_2 : J \to \mathbb{R}$ be indefinite HKS_α integrals of f with respect to G on J. Let F_1, F_2, G be regulated functions. Then there exists a constant $C \in \mathbb{R}$ such that $F_2 - F_1 \equiv C$ on J.*

Proof. By Proposition 5.7, F_1 and F_2 are HKS_α^∞ indefinite integrals on J. Consequently, by [8, Theorem 3.5], there is a constant $C \in \mathbb{R}$ such that $F_2 - F_1 = C$ holds almost everywhere on J. Further, since by Corollary 5.18 the functions F_1, F_2 are regulated the function $F_2 - F_1$ is obviously also regulated.

Now, let $x \in J$ be given such that $[x, x + \Delta) subset J$ for some $\Delta > 0$. Let I_n, $n \in \mathbb{N}$ be an arbitrary system of subintervals in $(x, x + \Delta)$ such that

$$x \in I_{n+1} \subsetneqq I_n \quad \text{for all} \;\; n \in \mathbb{N}.$$

Since $F_2 - F_1 = C$ a.e. on J, we can for any $n \in \mathbb{N}$ find a point $x_n \in I_n \setminus I_{n+1} J$ such that $F_2(x_n) - F_1(x_n) = C$. Obviously $\lim_{n \to +\infty} x_n = x$ and

$$\lim_{t \to x+} (F_2(t) - F_1(t)) = \lim_{n \to +\infty} (F_2(x_n) - F_1(x_n)) = C.$$

Analogously, we can prove that

$$\lim_{t \to x-} (F_2(t) - F_1(t)) = C.$$

Now, let $I = [a, b]$ be an arbitrary closed subinterval of J. Then, by Corollary 5.18, we have

$$F_2(x) - F_1(x) = \lim_{t \to x+} (F_2(t) - F_1(t)) = C \quad \text{for all} \;\; x \in [a, b).$$

Similarly,

$$F_2(x) - F_1(x) = \lim_{t \to x-} (F_2(t) - F_1(t)) = C \quad \text{for all} \;\; x \in (a, b].$$

The proof of our assertion follows immediately. □

Next definition is justified by Propositions 5.15 and 5.20.

Definition 5.21 (Definite HKS_α integral)**.** Let F, G, $f : [a,b] \to \mathbb{R}$ be measurable functions and $\alpha \geq 1$. Let F be the indefinite HKS_α integral of f with respect to G on $[a,b]$. Then for $\alpha = 1$, we define the definite integral as

$$\int_a^b f \, \mathrm{d}G := F(b) - F(a). \tag{5.7}$$

If F and G are regulated functions on $[a,b]$, then relation (5.7) defines the definite HKS_α integral also for $\alpha > 1$.

We close this paper by an analogue of the known Hake Theorem.

Proposition 5.22. *Let F be the indefinite HKS_α integral of the function f with respect to G on the interval $[a,b)$ and let there exist a proper limit*

$$L = \lim_{x \to b-} \big(F(x) - f(b)\,(G(x) - G(b))\big). \tag{5.8}$$

Then, if we put $F(b) = L$, F becomes the indefinite HKS_α integral of f with respect to G on $[a,b]$.

Proof. Let $\varepsilon > 0$ be given and $F(b) = L$.. In view of (5.8), there is a $\Delta > 0$ such that

$$\big|F(x) - F(b) - f(b)\,[G(x) - G(b)]\big| < \frac{\varepsilon}{2} \quad \text{for all} \quad x \in [b - \Delta, b].$$

Thus,

$$\sup_{x \in [b-\Delta,b]} \big|F(x) - F(b) - f(b)\,[G(x) - G(b)]\big| < \frac{\varepsilon}{2}$$

and, for each $z \in [b - \Delta, b]$,

$$\begin{aligned}
&\operatorname{osc}(F - f(b)\,G, [z,b]) \leq \operatorname{osc}(F - f(b)\,G, [b-\Delta,b]) \\
&\quad = \frac{1}{2} \sup_{x,y \in [b-\Delta,b]} \big|F(x) - F(b) - f(b)\,[G(x) - G(b)] - \big(F(y) - F(b) - f(b)\,[G(y) - G(b)]\big)\big| \\
&\quad \leq \frac{1}{2} \sup_{x \in [b-\Delta,b]} \big|F(x) - F(b) - f(b)\,[G(x) - G(b)]\big| + \frac{1}{2} \sup_{y \in [b-\Delta,b]} \big|F(y) - F(b) - f(b)\,[G(y) - G(b)]\big| \\
&\quad < \frac{\varepsilon}{2},
\end{aligned}$$

i.e.

$$\operatorname{osc}(F - f(b)\,G, [z,b]) < \frac{\varepsilon}{2} \quad \text{for each } z \in [b - \Delta, b]. \tag{5.9}$$

Further, let δ'_ε be a gauge on $[a,b)$ such that

$$\sum_{i=1}^n \operatorname{osc}\big(F - f(x_i)\,G, [a_i,b_i]\big) < \frac{\varepsilon}{2} \quad \text{for all} \quad \delta'_\varepsilon\text{-fine } \alpha\text{-systems } \{[a_i,b_i],x_i\}_{i=1}^n \quad \text{in } [a,b) \tag{5.10}$$

and let

$$\delta_\varepsilon(x) = \begin{cases} \min\{\delta_\varepsilon(x), \frac{1}{2}(b-x)\} & \text{if} \quad x \in [a,b), \\ \Delta & \text{if} \quad x = b. \end{cases}$$

Then, if the α-system $\{[a_i,b_i],x_i\}_{i=1}^n$ is δ_ε-fine in $[a,b]$ and $b \in \bigcup_{i=1}^n [a_i,b_i]$, its subsystem $\{[a_i,b_i],x_i\}_{i=1}^{n-1}$ is δ'_ε-fine α-system in $[a,b)$ and $x_n = b = b_n$ and $a_n > b - \Delta$. Hence, according to (5.9) and (5.10), for any δ_ε-fine system $\{[a_i,b_i],x_i\}_{i=1}^n$ in $[a,b]$ we have

$$\sum_{i=1}^n \operatorname{osc}(F - f(x_i)\,G, [a_i,b_i]) = \sum_{i=1}^{n-1} \operatorname{osc}(F - f(x_i)\,G, [a_i,b_i]) + \operatorname{osc}(F - f(b)\,G, [a_n,b_n]) < \varepsilon.$$

Of course, it is easy to derive the inequality

$$\sum_{i=1}^{n} \operatorname{osc}(F - f(x_i)\, G, [a_i, b_i]) < \varepsilon$$

also for the case that $b \notin \bigcup_{i=1}^{n} [a_i, b_i]$. This completes the proof. □

Remark 5.23. An analogous statement holds for the interval $(a, b]$ and for the limit at the point a from the right.

Open problems.

- By Proposition 5.9, the inclusion $HKS_\alpha^C \subseteqq HKS_\alpha^\infty$ is true for any $\alpha \geq 1$ and any integrator G. By [6, Example 3.1.2], if functions f, G, $F : [-1, 1] \to \mathbb{R}$ are given by

$$f(x) \equiv 1,\ G(x) = \begin{cases} x & \text{if } x \in [-1, 1] \setminus \{1/2k,\ k \in \mathbb{N}\}, \\ \frac{1}{x} & \text{if } x \in \{1/2k,\ k \in \mathbb{N}\} \end{cases} \quad \text{and} \quad F(x) = x,$$

  then F is a HKS_α^∞-indefinite integral of f with respect to G, but it is not a HKS_α^C-indefinite integral of f with respect to G. On the other hand, e.g. the function $F = G$ is obviously a HKS_α^C-indefinite integral of f with respect to G. However, an open problem remains whether it is possible to find couple of functions f, G which has HKS_α^∞-indefinite integral, while the corresponding HKS_α^C-indefinite integral does not exist.

- If $\alpha > 1$, G is regulated and the indefinite HKS_α-integral F exists and is also regulated, then by Proposition 5.20 F is unique up to a constant. Open question is whether the regulatedness assumption is really necessary for such a uniqueness statement.

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