RMF – 5. sada, 25.10.2024

1) Definice: Řekneme, že f_n konvergují slabě k f v $\mathcal{D}'(\mathbb{R})$, pokud pro každou $\varphi \in \mathcal{D}(\mathbb{R})$ platí, že $(f_n, \varphi) \xrightarrow[n\to\infty]{} (f, \varphi)$.

1a) $$(f_n, \varphi) = \int_{\mathbb{R}} \frac{4n^3+3n+1}{n^4x^2+n^2} \cdot \varphi(x)\, dx = \int_{\mathbb{R}} \frac{4n^3+3n+1}{n^2(t^2+1)} \cdot \varphi\left(\frac{t}{n}\right) \frac{dt}{n} =$$

$$\begin{vmatrix} nx = t \\ n\,dx = dt \end{vmatrix}$$

$$= \int_{\mathbb{R}} \underbrace{\left(4 + \frac{3}{n^2} + \frac{1}{n^3}\right) \cdot \frac{1}{t^2+1} \varphi\left(\frac{t}{n}\right)}_{=: g_n(t)} dt \xrightarrow[\text{VĚTA}]{\text{LEBESGUEOVA}} 4\varphi(0) \cdot \underbrace{\int_{\mathbb{R}} \frac{1}{t^2+1} dt}_{=\pi}$$

$$\forall t \in \mathbb{R}\ \forall n \in \mathbb{N}: \ |g_n(t)| \le 8 \cdot \max_{x\in\mathbb{R}} |\varphi(x)| \cdot \frac{1}{t^2+1} \in L^1(\mathbb{R})$$

tedy $(f_n, \varphi) \xrightarrow[n\to\infty]{} 4\pi \cdot \varphi(0) = (4\pi \cdot \delta, \varphi)$

$f_n(0) = 4n + \frac{3}{n} + \frac{1}{n^2} \xrightarrow[n\to\infty]{} \infty \Rightarrow f_n$ nekonvergují v klasickém smyslu

1b) $$(f_n, \varphi) = \int_{\mathbb{R}} \frac{n}{n^4x^2+1} \varphi(x)\, dx = \int_{\mathbb{R}} \frac{1}{n^2t^2+1} \varphi\left(\frac{t}{n}\right) dt$$

$$\begin{vmatrix} nx = t \\ n\,dx = dt \end{vmatrix}$$

$$\xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \lim_{n\to\infty} \left(\frac{1}{n^2t^2+1} \cdot \varphi\left(\frac{t}{n}\right)\right) dt \overset{\text{s.v.}}{=} \int_{\mathbb{R}} 0 \cdot \varphi(0)\, dt = 0 = (0, \varphi)$$

LEBESGUEOVA VĚTA: $\forall t \in \mathbb{R}\ \forall n \in \mathbb{N}: \left|\frac{\varphi\left(\frac{t}{n}\right)}{n^2t^2+1}\right| \le \underbrace{\max_{x\in\mathbb{R}} |\varphi(x)| \cdot \frac{1}{t^2+1}}_{\in L^1(\mathbb{R})}$

tedy $(f_n, \varphi) \xrightarrow[n\to\infty]{} 0$, ale $f_n(0) = n \xrightarrow[n\to\infty]{} \infty$

tedy nemají limitu v klasickém smyslu

1c) $(f_n, \varphi) = \int_{\mathbb{R}} \frac{\operatorname{arccotg}(nx)}{n^2x^2+1}\,\varphi(x)\,dx = \int_{\mathbb{R}} \underbrace{\frac{\operatorname{arccotg} t}{t^2+1}\,\varphi\left(\frac{t}{n}\right)\frac{1}{n}}_{=:\, g_n(t)}\,dt$

$\left|\begin{array}{l} nx = t \\ n\,dx = dt \end{array}\right|$ $dx = \frac{1}{n}\,dt$

LEBESGUEOVA VĚTA

$\xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \underbrace{\lim_{n\to\infty}(\ )}_{0}\,dt = \int_{\mathbb{R}} 0\,dt = \underline{\underline{0}}$

$\forall n\in\mathbb{N}\ \forall t\in\mathbb{R}: |g_n(t)| \le \underbrace{\max_{x\in\mathbb{R}}|\varphi(x)|}_{<+\infty}\cdot\frac{\operatorname{arccotg} t}{t^2+1} \in L^1(\mathbb{R})$

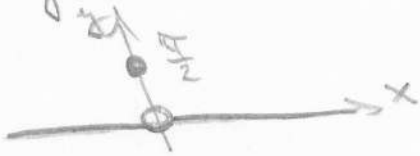
$\int_{\mathbb{R}} \frac{\operatorname{arccotg} t}{t^2+1}\,dt = -\int_{\pi}^{0} z\,dz = \int_0^{\pi} z\,dz = \left[\frac{z^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}$

$\left|\begin{array}{l} \operatorname{arccotg} t = z \\ -\frac{1}{t^2+1}\,dt = dz \end{array}\right|$

V klasickém smyslu: $\forall x\in\mathbb{R}\setminus\{0\}: \lim_{n\to\infty} f_n(x) = 0$

$\wedge\ \forall n\in\mathbb{N}: f_n(0) = \operatorname{arccotg} 0 = \frac{\pi}{2}$

$f_n(x) \xrightarrow[n\to\infty]{} f(x) = \begin{cases} 0, & x\neq 0 \\ \frac{\pi}{2}, & x=0 \end{cases}$



ale $f_n(x) \xrightarrow[n\to\infty]{} 0$ v $\mathcal{D}'(\mathbb{R})$

1d) $(f_n, \varphi) = \int_{\mathbb{R}} \frac{n\operatorname{arccotg}(nx)}{n^2x^2+1}\,\varphi(x)\,dx = \int_{\mathbb{R}} \frac{\operatorname{arccotg} t}{t^2+1}\cdot\varphi\left(\frac{t}{n}\right)dt$

$\left|\begin{array}{l} nx = t \\ n\,dx = dt \end{array}\right|$

$\xrightarrow[n\to\infty]{} \varphi(0)\cdot\frac{\pi^2}{2} = \underline{\underline{\left(\frac{\pi^2}{2}\cdot\delta, \varphi\right)}}$, ale v klasickém smyslu nemá limitu neboť $f_n(0) = \frac{\pi}{2}n \xrightarrow[n\to\infty]{} \infty$

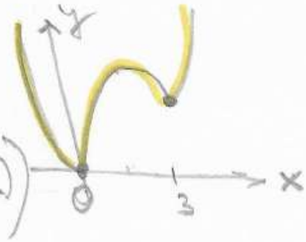
LEB. VĚTA

$|\ \ | \le \max_{x\in\mathbb{R}}|\varphi(x)|\cdot\frac{\operatorname{arccotg} t}{t^2+1} \in L^1(\mathbb{R})$

2) <u>Věta</u>: $$\overline{f}' = \{f'\} + \sum_{x_k \in M} \delta(x-x_k)\cdot\left(f(x_{k+}) - f(x_{k-})\right)$$

pro funkci $f \in C^1(\mathbb{R}\setminus M)$, kde M je konečná nebo spočetná množina bez hromadného bodu a $f' \in L^1_{loc}(\mathbb{R})$, $\{f'\}$ je klasická derivace tam, kde to lze; tj. na $\mathbb{R}\setminus M$

2a) $f(x) = |x| + |2x^2-6x| = |x| + 2|x(x-3)|$; $M = \{0;3\}$



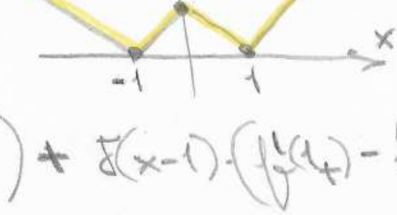
$$f'(x) = \operatorname{sgn} x + (4x-6)\cdot\operatorname{sgn}(2x^2-6x)$$

$$f''(x) = 4\operatorname{sgn}(2x^2-6x) + \delta(x)\cdot\left(f'(0_+) - f'(0_-)\right) + \delta(x-3)\cdot\left(f'(3_+) - f'(3_-)\right)$$

$$= 4\operatorname{sgn}(2x^2-6x) + \delta(x)\cdot\underbrace{\left(1+6-(-1-6)\right)}_{14} + \delta(x-3)\cdot\underbrace{\left(1+6-(1-6)\right)}_{12}$$

$$= 4\operatorname{sgn}(2x(x-3)) + 14\delta(x) + 12\delta(x-3)$$

2b) $f(x) = \left||x|-1\right|$, $M = \{0,1,-1\}$

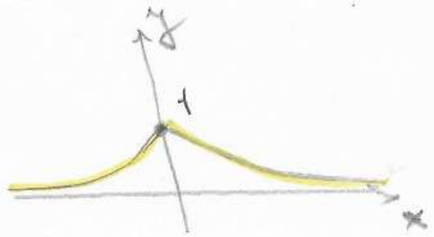


$$f'(x) = \operatorname{sgn} x \cdot \operatorname{sgn}(|x|-1)$$

$$f''(x) = \{0\} + \delta(x)\cdot\underbrace{\left(f'(0_+) - f'(0_-)\right)}_{(-1)-1} + \delta(x-1)\cdot\underbrace{\left(f'(1_+) - f'(1_-)\right)}_{1-(-1)}$$

$$+ \delta(x+1)\cdot\underbrace{\left(f'(-1_+) - f'(-1_-)\right)}_{1-(-1)} = -2\delta(x) + 2\delta(x-1) + 2\delta(x+1)$$

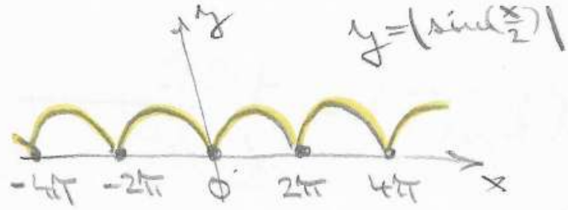
2c) $f(x) = \frac{1}{1+|x|}$ $\qquad M = \{0\}$



$$f'(x) = \operatorname{sgn} x \cdot \left(-\frac{1}{(1+|x|)^2}\right)$$

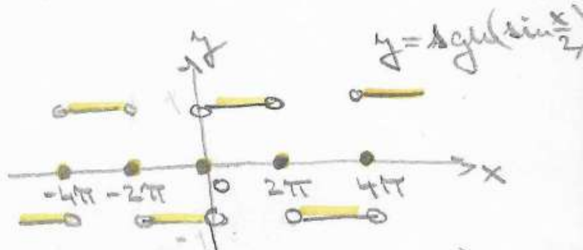
$$f''(x) = \underbrace{(\operatorname{sgn} x)^2}_{=1 \text{ na } \mathbb{R}\setminus\{0\}} \cdot \frac{2}{(1+|x|)^3} + \delta(x)\cdot \underbrace{\left(f'(0_+) - f'(0_-)\right)}_{(-1)-1}$$

$$= \frac{2}{(1+|x|)^3} - 2\cdot\delta(x)$$



2d) $f(x) = \left|\sin\left|\frac{x}{2}\right|\right| = \left|\sin\left(\frac{x}{2}\right)\right|$

$M = \{2k\pi;\ k\in\mathbb{Z}\}$



$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)\cdot \operatorname{sgn}\left(\sin\left(\frac{x}{2}\right)\right)$$

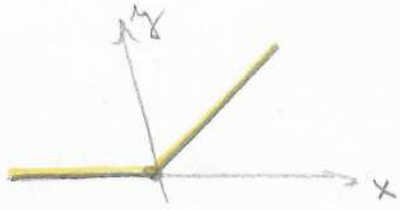
$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)\cdot\operatorname{sgn}\left(\sin\left(\frac{x}{2}\right)\right)$$

$$+ \sum_{k\in\mathbb{Z}} \delta(x-2k\pi)\cdot \underbrace{\left(f'(2k\pi_+) - f'(2k\pi_-)\right)}_{\frac{1}{2}\cdot \underbrace{\cos(k\pi)}_{(-1)^k}\cdot 2\cdot(-1)^k}$$

$$= -\frac{1}{4}\cdot\left|\sin\left(\frac{x}{2}\right)\right| + \sum_{k\in\mathbb{Z}} \delta(x-2k\pi)$$

„velikost skoku" funkce $\operatorname{sgn}(\sin\frac{x}{2})$ v bodech $2k\pi$, $k\in\mathbb{Z}$ je $2\cdot(-1)^k$

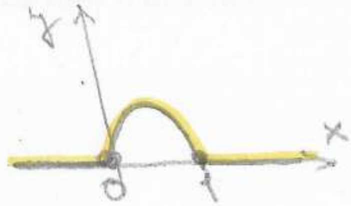
2e) $f(x) = |x| \cdot \Theta(x)$, $M = \{0\}$



$$f'(x) = \operatorname{sgn} x \cdot \Theta(x)$$

$$f''(x) = \{0\} + \delta(x) \cdot \big(\underbrace{f'(0_+) - f'(0_-)}_{1 - 0} \big) = \underline{\underline{\delta(x)}}$$

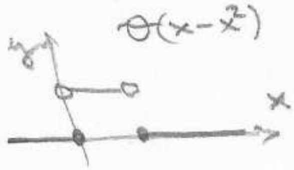
2f) $f(x) = (x - x^2) \cdot \Theta(x - x^2)$



$M = \{0, 1\}$

$$\Theta(x - x^2) = \begin{cases} 1, & x \in (0,1) \\ 0, & x \notin (0,1) \end{cases}$$

$\Theta(x - x^2)$



$$f'(x) = (1 - 2x) \cdot \Theta(x - x^2)$$

$$f''(x) = (-2) \cdot \Theta(x - x^2) + \delta(x) \cdot \big(\underbrace{f'(0_+) - f'(0_-)}_{1 - 0} \big)$$

$$+ \delta(x-1) \cdot \big(\underbrace{f'(1_+) - f'(1_-)}_{0 - 1 \cdot (-1)} \big)$$

$$= (-2) \cdot \chi_{(0,1)}(x) + \delta(x) + \delta(x-1)$$