(1.) X ... doba životnosti (o měsících). 2,5 roku = 30 měs.

$X \sim Exp\left(\frac{1}{30}\right)$

$$f_X(x) = \begin{cases} \frac{1}{30} e^{-\frac{1}{30}x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

a) $P[X \geq 15] = 1 - P[X < 15] = 1 - \int_0^{15} \frac{1}{30} e^{-\frac{x}{30}} dx =$

$= 1 - \left[-e^{-\frac{x}{30}}\right]_0^{15} = 1 - \left(-e^{-\frac{1}{2}} + 1\right) = e^{-\frac{1}{2}} = \underline{\underline{\frac{1}{\sqrt{e}} \doteq 60\%}}$

b) $P[X \leq x] = 90\%$

$\int_0^x \frac{1}{30} e^{-\frac{t}{30}} dt = \left[-e^{-\frac{t}{30}}\right]_0^x = -e^{-\frac{x}{30}} + 1 = 0{,}9$

$+ e^{-\frac{x}{30}} = +0{,}1$

$-\frac{x}{30} = \ln 0{,}1$

$\underline{\underline{x \doteq 69 \text{ (měsíců)}}}$.

(2.) $f(x,y) = \begin{cases} \sqrt[3]{x} + xy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{jinak} \end{cases}$

$f_X(x) = \int_R f(x,y)\,dy = \int_0^1 \left(\sqrt[3]{x} + xy\right) dy = \left[\sqrt[3]{x}\,y + x\frac{y^2}{2}\right]_{y=0}^{y=1} =$

$= \sqrt[3]{x} + \frac{1}{2}x, \quad x \in \langle 0,1 \rangle$

$f_Y(y) = \int_R f(x,y)\,dx = \int_0^1 \left(\sqrt[3]{x} + xy\right) dx = \left[\frac{3}{4}x^{\frac{4}{3}} + \frac{x^2}{2}y\right]_{x=0}^{x=1} =$

$= \frac{3}{4} + \frac{1}{2}y, \quad y \in \langle 0,1 \rangle$

$f(x,y) \neq f_X(x) \cdot f_Y(y) \Rightarrow X, Y$ jsou závislé

Střední hodnoty: $EX = \int_R x f(x)\,dx = \int_0^1 x\left(\sqrt[3]{x} + \frac{1}{2}x\right)dx =$

$= \int_0^1 \left(x^{\frac{4}{3}} + \frac{1}{2}x^2\right)dx = \left[\frac{3}{7}x^{\frac{7}{3}} + \frac{x^3}{6}\right]_0^1 = \frac{3}{7} + \frac{1}{6} = \frac{25}{42}$

$EY = \int_R y f(y)\,dy = \int_0^1 y\left(\frac{3}{4} + \frac{1}{2}y\right)dy = \int_0^1 \left(\frac{3}{4}y + \frac{1}{2}y^2\right)dy$

$= \left[\frac{3}{4}\frac{y^2}{2} + \frac{1}{2}\frac{y^3}{3}\right]_0^1 = \frac{3}{8} + \frac{1}{6} = \frac{9+4}{24} = \frac{13}{24}$

2. momenty: $EX^2 = \int_R x^2 f(x)\,dx = \int_0^1 x^2\cdot\left(\sqrt[3]{x} + \frac{1}{2}x\right)dx =$

$= \int_0^1 \left(x^{\frac{7}{3}} + \frac{1}{2}x^3\right)dx = \left[\frac{3}{10}x^{\frac{10}{3}} + \frac{x^4}{8}\right]_0^1 = \frac{3}{10} + \frac{1}{8} = \frac{12+5}{40} = \frac{17}{40}$

$EY^2 = \int_R y^2 f(y)\,dy = \int_0^1 y^2\cdot\left(\frac{3}{4} + \frac{1}{2}y\right)dy = \int_0^1 \left(\frac{3}{4}y^2 + \frac{1}{2}y^3\right)dy$

$= \left[\frac{3}{4}\cdot\frac{y^3}{3} + \frac{1}{2}\cdot\frac{y^4}{4}\right]_0^1 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

Rozptyly: $\operatorname{var} X = EX^2 - (EX)^2 = \frac{17}{40} - \left(\frac{25}{42}\right)^2 \doteq 0{,}070$

$\operatorname{var} Y = EY^2 - (EY)^2 = \frac{3}{8} - \left(\frac{13}{24}\right)^2 \doteq 0{,}081$

$E(XY) = \iint_{R^2} xy\,f(x,y)\,dy\,dx = \int_0^1\int_0^1 xy\left(\sqrt[3]{x} + xy\right)dy\,dx =$

$= \int_0^1\int_0^1 \left(x^{\frac{4}{3}}y + x^2y^2\right)dy\,dx = \int_0^1 \left[x^{\frac{4}{3}}\cdot\frac{y^2}{2} + x^2\cdot\frac{y^3}{3}\right]_0^1 dx =$

$$= \int_0^1 \left(\frac{1}{2} x^{\frac{4}{3}} + \frac{1}{3} x^2 \right) dx = \left[\frac{1}{2} \cdot \frac{3}{7} x^{\frac{7}{3}} + \frac{x^3}{9} \right]_0^1 =$$

$$= \frac{3}{14} + \frac{1}{9} = \frac{27+14}{126} = \frac{41}{126} \doteq 0{,}32$$

kovariance: $cov(X,Y) = E(XY) - EX \cdot EY =$

$$= \frac{41}{126} - \frac{25}{42} \cdot \frac{13}{24} = \frac{1}{336} \doteq 0{,}0029$$

korelace: $corr(X,Y) = \frac{cov(X,Y)}{\sqrt{varX} \cdot \sqrt{varY}} \doteq \frac{\frac{1}{336}}{\sqrt{0{,}07} \cdot \sqrt{0{,}081}}$

$$\doteq 0{,}039$$

kovariační matice: $Var(X,Y) = \begin{pmatrix} varX & cov(X,Y) \\ cov(Y,X) & varY \end{pmatrix}$

$$= \begin{pmatrix} 0{,}070 & 0{,}029 \\ 0{,}029 & 0{,}081 \end{pmatrix}$$

(3.) 13, 14, 22, 31, 50, 51

$P[X=1] = \frac{3}{6} = \frac{1}{2}$

$P[X=0] = \frac{3}{6} = \frac{1}{2}$

X... číslo je liché

$P[Y=4] = \frac{3}{6} = \frac{1}{2}$

$P[Y=5] = \frac{2}{6} = \frac{1}{3}$

$P[Y=6] = \frac{1}{6}$

Y... ciferný součet

X \ Y	4	5	6
0	$\frac{1}{6}$	$\frac{2}{6}$	0
1	$\frac{2}{6}$	0	$\frac{1}{6}$

$EX = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$

$EY = 4 \cdot \frac{1}{2} + 5 \cdot \frac{1}{3} + 6 \cdot \frac{1}{6} = \frac{14}{3}$

$E(XY) = \sum x_i y_j P[X=x_i, Y=y_j] = 1 \cdot 4 \cdot \frac{2}{6} + 1 \cdot 6 \cdot \frac{1}{6} = \frac{7}{3}$

kovariance: $cov(X,Y) = E(XY) - EX \cdot EY = \frac{7}{3} - \frac{1}{2} \cdot \frac{14}{3}$

$= \frac{1}{3} \cdot (7 - 7) = \underline{0} \Rightarrow cov(X,Y) = 0$

přesto jsou X, Y závislé, neboť

$P[X=1, Y=4] \neq P[X=1] \cdot P[Y=4]$

$\frac{1}{3} \qquad \frac{1}{2} \qquad \frac{1}{2}$

4. $X \sim N(100, 15^2)$

$$P[X > 140] = 1 - P[X \leq 140] =$$

$$= 1 - P\left[\frac{X-100}{15} \leq \frac{40}{15}\right] = 1 - \Phi\left(\frac{8}{3}\right)$$

$\frac{X-100}{15} \sim N(0,1)$

$$\doteq 1 - \Phi(2{,}66)$$

$$= 1 - 0{,}99609$$

$$\doteq \underline{0{,}00391}$$

$\Rightarrow$ Přibližně 0,4% populace je geniální.

5. $$P\left[t_{\frac{\alpha}{2},n-1} \leq \sqrt{n} \cdot \frac{\bar{X}_n - \mu}{S_n} \leq t_{1-\frac{\alpha}{2},n-1}\right] = 90\%$$

$\bar{X}_n = 2\,000\,000$ Kč

$n = 36$ měsíců

$S_n = 24\,000$ Kč

$1 - \alpha = 0{,}9$

$\alpha = 0{,}1$

$t_{1-\frac{\alpha}{2},n-1} = t_{0{,}95;35} = 1{,}69$

$$\Rightarrow P\left[\mu \in \left(2\,000\,000 \mp \underbrace{t_{0{,}95;35}}_{1{,}69} \cdot \frac{24\,000}{\sqrt{36}}\right)\right] = 90\%$$

$$\underbrace{1{,}69 \cdot \frac{24\,000}{6}}_{6760}$$

interval spolehlivosti: $(1\,993\,240,\ 2\,006\,760)$